LOGICAL GENERALITY OF APPLICABILITY OF FRACTIONAL DIFFERENTIAL EQUATION AMONG LAPLACE TRANSFORMS

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Abstract: Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. In recent years, considerable amount of research in fractional calculus was published in engineering and mathematical physics literature. Indeed, recent advances of fractional calculus are dominated by modern examples of applications in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, non-linear control theory, image processing, non-linear biological systems, astrophysics and electrochemistry. There is no doubt that fractional calculus has become an exciting new mathematical method of solution of diverse problems in science, engineering and applied mathematics. One of the main applications of the fractional calculus is modeling of the intermediate physical process. A very important model is the fractional differential and wave equations. Integral transformations have been successfully used for almost two centuries in solving many problems in applied mathematics, mathematical physics, and engineering science. Historically, the origin of the integral transforms including the Laplace and Fourier transforms can be traced back to celebrated work of Pieere – Simon – Laplace (1749 – 1827) on probability theory in the 1780s and to monumental treatise of Joseph Fourier (1768 –1830) on La TheorieAnalytique de la Chaleur published in 1822. Some of the recent and interesting application ns are as follows which shows the versatility of these transforms. J. Membrez et al have used the Laplace transform to determine protein adsorption on porous beads. G. B. Davis. used Laplace transform technique to find the analytical solution to single diffusion-convection equation over a finite domain. Li Renetal. applied it for solving convection dispersion equations. Fourier and Laplace transforms can be used in areas such as medical field for blood-velocity/time wave form over cardiac cycle from common femoral artery, in the analysis of functionally graded plates under thermo mechanical loading and in probability theory for the integral expression for positive part moments (p>0) of random variables.

I. INTRODUCTION

Fractional differential equations are a generalization of ordinary differential equations and integration to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation. It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations. However, in case of fractional-order integration and differentiation, which represent a rapidly growing field both in theory and in applications to real world problems, it is not so.

II. BASIC DEFINITIONS

This section is devoted to review three important definitions of fractional derivative and give
some examples of fractional differential equations equipped by them.

i. Riemann-Liouvill definition:
The popular definition of fractional derivative is this one:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}} $$
$$ (n-1 \le \alpha < n) $$

This operator has the following important properties:

For a function f ${}_aD_t^\alpha \; {}_aD_t^\beta f(x) = {}_a D_t^{\alpha+\beta}$

By using of this definition, V. V. Anh and R. Mcvinish considered fractional differential equations of the general form

$$ (A_n D^{\beta_n} + \cdots A_1 D^{\beta_1} + A_0 D^{\beta_0}) X(t) = \dot{L}(t) $$
$$ \beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0, \quad n \ge 1 $$

where ˙L is Levy noise.

Fractional differential equations in terms of the Riemann-Liouvill derivatives require initial conditions expressed in terms of initial values of fractional derivatives of the unknown function.

For example, in the following initial value problem (where n−1 < α < n):

ii. Grunwald-Letnikove:
This is another joined definition which is sometimes useful.

$$ {}_aD_t^\alpha f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh) $$

III. LAPLACE TRANSFORM

The Laplace transform has a long history, dating back to L. Euler's paper 'De Constructione Aequationum' from 1737. Since then it has been widely used in mathematics, in particular in ordinary differential, difference and functional equations. An informative description of the contributions of mathematicians like Euler, Lagrange, Laplace, Fourier,Poisson, Cauchy, Abel, Liouville, Boole, Riemann, Pincherle, Amaldi, Tricomi, Picard,Mellin, Borel, Heaviside, Bateman, Titchmarsh, Bernstein, Doetsch, Widder and many others can be found in two historical surveys by M. Deakin (1981, 1982).

Ordinary and partial differential equations describe the way certain quantities vary with time,such as the current in an electrical circuit, the oscillations of a vibrating membrane, or the flow of heat through an insulated conductor. These equations are generally coupled with initial conditions that describe the state of the system at time t=0.A very powerful technique for solving these problems is that of the Laplace transform, which literally transforms the original differential equation into an elementary algebraic expression. This latter can then simply be transformed once again, into the solution of the original problem. This technique is known as the "Laplace transform method."

Basic Definition of Laplace Transform

If f(t) is defined for t≥0 the (unilateral) Laplace transform (Pierre-Simon Laplace) and its inverse are defined by:

$$\mathcal{L}: \quad f(t) \mapsto F(s) \quad = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)dt,$$

$$\mathcal{L}^{-1}: \quad F(s) \mapsto f(t) \quad = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}ds.$$

Note that if f(t) as then the first integral converges for all complex numbers with real part greater than s0, and in the second integral we then demand that .

Whenever the limit exists.When it does, the integral is said to converge. If the limit does not exist, the integral is said to diverge and there is no Laplace transform defined for f. The notation will also be used to denote the Laplace transform of f, and the integral is the ordinary Riemann (improper) integral.

Then the one-sided Laplace renovate

$$\mathcal{L}^{-}[g(t); s] = \int_{-\infty}^{0} g(t)e^{-st}dt$$
$$= \int_0^\infty g(-t)e^{st}dt$$

Converges enormously and is methodical in the left half plane Re(s) <β.

Under the same state of affairs on g(t) and if β > α the two-sided Laplace transform.

$$\mathcal{L}[g(t); s] = \int_{-\infty}^{\infty} g(t)e^{-st}dt$$

Converges enormously and is methodical in the vertical strip α <<Re(s) < β.

Now if we let t =- log x and g(-log x) = f (x) then e-st = es log x = xs.

Hence,

$$\mathcal{L}[g(t); s] = \int_{-\infty}^{\infty} g(t)e^{-st}dt$$
$$= -\int_{\infty}^{0} x^{s-1}f(x)dx$$
$$= \mathcal{M}[f(x); s].$$

So the Mellin makeover of f (x) is the two-sided Laplace renovate of g(t) where t = - log x and it converges absolutely and is analytic in the vertical strip α <<Re(s) < β.

Now since g(t) = O(e_αt) as t →∞ we obtain f (x) = O(x-α) as x →0+.

Also g(t) = O(e_βt) as t →-∞ implies f (x) = O(x-β) as x →∞.

Summing out of cradle we contain proved the subsequent lemma:

Lemma 1:

The conditions f (x) = O(x-α) as x →0+ and f (x) = O(x-β) as x →∞ where α< β guarantee that f *(s) exists in the strip h{α,β}

Hence monomials xc, including constants do not have Mellin transforms.

Example 1.1

The function of *f*(x)= e-x = O(x0) as x→0+ and e-x = O(x-b) as x→∞ for any b>0 so that its transform

$$f^*(s) = \int_0^\infty e^{-x}x^{s-1}dx = \Gamma(s)$$

is defined and analytic on {0,∞}

Example 1.2

The function *f*(x)= (ex – 1)-1 satisfies *f*(x)= O(x-1) as x→O+ and *f*(x)= O(x-b) for all b>0 as x→∞.Hence *f*(x) is analytic and defined on {1,∞}.we find

$$f^*(s) = \int_0^\infty \frac{1}{1-e^{-x}}x^{s-1}dx$$
$$= \int_0^\infty \frac{e^{-x}}{(1-e^{-x})}x^{s-1}dx$$
$$= \int_0^\infty \sum_{n=1}^{\infty} e^{-nx}x^{s-1}dx$$
$$= \sum_{n=1}^{\infty} \int_0^\infty e^{-nx}x^{s-1}dx$$
$$= \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s}$$
$$= \Gamma(s)\zeta(s).$$

We require that R <(s) > 1 for convergence of the Riemann-zeta function and we see that this validates the strip {1,∞} on which f *(s) is defined and analytic.

Example 1.3

The function f (x) = (1 + x)-1 is O(x0) as x →0+ and O(x-1) as x →∞. Hence a guaranteed strip of existence for f *(s) is {0,1} .Set x = t /1-t.then

$$\begin{aligned} f^*(s) &= \int_0^1 \left(\frac{t}{1-t}\right)^{s-1} \frac{1}{1+\frac{t}{1-t}}(1-t)^{-2}dt \\ &= \int_0^1 \left(\frac{t}{1-t}\right)^{s-1} (1-t)^{-1}dt \\ &= \int_0^1 t^{s-1}(1-t)^{-s}dt \\ &= \beta(s, 1-s) \\ &= \Gamma(s)\Gamma(1-s). \end{aligned}$$

We can also evaluate the Mellin transform of f (x) using complex analysis. Consider the branch of the function zx-1/z+1 defined on the slit plane C/[0;∞) by

$$f(z) = \frac{r^{s-1}e^{i(s-1)\theta}}{z+1}; \quad z = re^{i\theta}, \quad 0 < \theta < 2\pi.$$

The values on the bottom edge are obtained from those on the top edge by multiplying by the phase factor e2πi(s-1).

$$f(z) = \frac{r^{s-1}e^{i(s-1)\theta}}{z+1}; \quad z = re^{i\theta}, \quad 0 < \theta < 2\pi.$$

For €> 0 small and R > 0 large, we consider the keyhole domain D (see Figure 1 in the appendix) consisting of z in the slit plane C/[0,∞) satisfying <IzI< R:f (z) has a pole in D, a simple pole at z = -1, with residue

$$\begin{aligned} \text{Res}\left[\frac{z^{s-1}}{z+1}, -1\right] &= \lim_{z\to -1}(z+1)\frac{z^{s-1}}{z+1} \\ &= \lim_{z\to -1}\frac{z^s}{z} \\ &= -e^{\pi i s} \end{aligned}$$

$$f(z) = \frac{r^{s-1}e^{i(s-1)\theta}}{z+1}; \quad z = re^{i\theta}, \quad 0 < \theta < 2\pi.$$

Hence the residue theorem yields

$$\int_{\partial D} f(z)dz = -2\pi i e^{\pi i s}$$

The integral around δD breaks into the sum of 4 integrals.

$$\Longrightarrow -2\pi i e^{\pi i s} = \int_\epsilon^R \frac{x^{s-1}}{1+x}dx + \int_{\Gamma_R} \frac{z^{s-1}}{z+1}dz + \int_R^\epsilon \frac{e^{2\pi i(s-1)}x^{s-1}}{1+x}dx + \int_{\Gamma_\epsilon} \frac{z^{s-1}}{z+1}dz$$

For the integrals over ⅂R and ⅂€ we have

$$\left|\int_{\Gamma_R} \frac{z^{s-1}}{z+1}dz\right| \le \frac{R^{s-1}}{R-1}2\pi R = O\left(R^{s-1}\right)$$

$$\left|\int_{\Gamma_\epsilon} \frac{z^{s-1}}{1+z}dz\right| \le \frac{\epsilon^{s-1}}{1-\epsilon}2\pi\epsilon = O\left(\epsilon^s\right)$$

Since 0 < s < 1 both these integrals vanish as R ! 1 and _ ! 0.

$$\Longrightarrow -2\pi i e^{\pi i s} = \left(1 - e^{2\pi i(s-1)}\right)\int_0^\infty \frac{x^{s-1}}{1+x}dx$$

$$\begin{aligned} \Longrightarrow \int_0^\infty \frac{x^{s-1}}{1+x}dx &= \frac{-2\pi i e^{\pi i s}}{1-e^{2\pi i(s-1)}} \\ &= \frac{2\pi i}{e^{\pi i s - 2\pi i} - e^{-\pi i s}} \\ &= \frac{\pi}{\sin(\pi s)} \end{aligned}$$

Hence $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$.

Example 1.4

Suppose f (x) = (1 + x)-n. Setting t = x/1+x we obtain

$$\begin{aligned} f^*(s) &= \int_0^1 t^{s-1}(1-t)^{n-s-1}dt \\ &= \beta(s, n-s) \\ &= \frac{\Gamma(s)\Gamma(n-s)}{\Gamma(n)}. \end{aligned}$$

Example 1.5

Suppose f (t) = sin t = i (eit – e-it): Consider the region D as in Figure 2 of the appendix.Since e-t ts-1 is analytic in D we have by Cauchy's theorem:

$$0 = \oint_{\partial D} e^{-t}t^{s-1}dt$$

$$= \int_\epsilon^R e^{-u}u^{s-1}du + i\int_0^R e^{-(R+iu)}(R+iu)^{s-1}du - \int_0^R e^{-(u+iR)}(u+iR)^{s-1}du$$

$$-i\int_\epsilon^R e^{-iu}(iu)^{s-1}du - \int_0^{\frac{\pi}{2}} e^{-\epsilon e^{i\theta}}(\epsilon e^{i\theta})^{s-1}i\epsilon e^{i\theta}d\theta$$

Letting €→0 and R →∞we have:

$$\int_0^{\frac{\pi}{2}} \left|e^{-\epsilon e^{i\theta}}(\epsilon e^{i\theta})^{s-1}i\epsilon e^{i\theta}\right|d\theta = O(\epsilon^s) \to 0 \quad if \quad \text{Res}(s) > 0$$

$$\int_0^R \left|e^{-(R+iu)}(R+iu)^{s-1}\right|du = O\left(e^{-R}R^{Re(s-1)}R\right) \to 0$$

$$\int_0^R \left|e^{-(u+iR)}(u+iR)^{s-1}\right|du = O\left(R^{Re(s-1)}\int_0^R e^{-u}du\right) \to 0 \quad if \quad \text{Res}(s) < 1$$

$$\Longrightarrow \int_0^\infty e^{-u}u^{s-1}du = i^s\int_0^\infty e^{-iu}u^{s-1}du$$

$$\Longrightarrow \int_0^\infty e^{-iu}u^{s-1}du = i^{-s}\int_0^\infty e^{-u}u^{s-1}du = e^{-s\log i}\Gamma(s) = e^{-\frac{i\pi s}{2}}\Gamma(s)$$

Likewise replacing i with -i in the previous equation we obtain $\int_0^\infty e^{iu}u^{s-1}du = e^{\frac{i\pi s}{2}}\Gamma(s).$ Combining the two we get

$$\int_0^\infty (\sin t)t^{s-1}dt = \frac{1}{2i}\left(e^{-\frac{i\pi s}{2}} - e^{\frac{i\pi s}{2}}\right) = \sin\left(\frac{\pi s}{2}\right)\Gamma(s).$$

We proceed to look at some functional properties of the Mellin transform.

Theorem :

Let f (x) be a function whose transform admits the fundamental strip

H{α,β} Let p,µ and ν be positive real numbers. Then the following relations hold:

$$(i)\quad \mathcal{M}[f(\mu x); s] = \mu^{-s} f^*(s) \quad s \in \langle \alpha, \beta \rangle$$

$$(ii)\quad \mathcal{M}\left[\sum_{k\in A} \lambda_k f(\mu_k x); s\right] = \left(\sum_{k\in A} \lambda_k \mu_k^{-s}\right) f^*(s) \quad ,\mu_k > 0$$

$$(iii)\quad \mathcal{M}[x^\nu f(x); s] = f^*(s+\nu) \quad s \in \langle \alpha - \nu, \beta - \nu \rangle$$

$$(iv)\quad \mathcal{M}[f(x^\rho); s] = \frac{1}{\rho} f^*\left(\frac{s}{\rho}\right) \quad s \in \langle \rho\alpha, \rho\beta \rangle$$

Proof:

$$(i)\quad \mathcal{M}[f(\mu x); s] = \int_0^\infty x^{s-1} f(\mu x) dx = \int_0^\infty f(t)\left(\frac{t}{\mu}\right)^{s-1} \frac{dt}{\mu} = \mu^{-s} f^*(s).$$

(ii) Since the Mellin transform is linear this follows immediately from (i):

$$(iii)\quad \mathcal{M}[x^\nu f(x); s] = \int_0^\infty f(x) x^{s+\nu-1} dx = f^*(s+\nu).$$

We have for example from Theorem 2.1(iv)M[e-x2 ; s] = 1/2Ţ(s/2) on <0,∞> with f (x) = e-x and ῥ = 2. Wanting to expand our range of Mellin transforms we find by differentiation under the integral sign:

$$\frac{d}{ds} f^*(s) = \mathcal{M}[f(x)\log x; s]$$

For instance the transform of e-x log x is Ţ(s). If we want to transform the derivative of a function integration by parts yields:

$$\int_0^\infty f'(x)x^{s-1}dx = \left[f(x)x^{s-1}\right]_0^\infty - (s-1)\int_0^\infty f(x)x^{s-2}dx$$

The term $\left|f(x)x^{s-1}\right|_0^\infty$ equals 0 since we assume the reason why f*(s) exists for s € {α,β}is because $\lim_{x\to 0} x^\rho f(x) = 0$ for $\Re(s) > \alpha$ and $\lim_{x\to\infty} x^\rho f(x) = 0$ for $\Re(s) < \beta$.

Thus,

$$\mathcal{M}[f'(x); s] = -(s-1)f^*(s-1).$$

For instance from Example we can derive the Mellin transform of f (x) = log(1 + x)

Special Cases

Laplace Transform Method Solution of Fractional Ordinary Differential Equations

Summary

Eltayeb. A.M. Yousif, et.al has proposed Laplace transform method for solving the fractional ordinary differential equations with constant and variable coefficients.

The solutions are expressed in terms of Mittage-Leffller functions, and then written in a compact simplified form.

As special case, when the order of the derivative is two the result is simplified to that of second order equation.

Findings

The Laplace transformation method has been successfully applied to find an exact solution of fractional ordinary differential equations, with constant and variable coefficients.

Some theorems were introduced; also special formulas of Mittage-Leffler function were derived with their proofs.

The method was applied in a direct way without using any assumptions.

The results showed that the Laplace transformation method needs small size of computations compared to the Adomain decomposition method (ADM), variation iteration method (VIM) and homotopy perturbation method (HPM).

It was concluded that the Laplace transformation method is a powerful, efficient and reliable tool for the solution of fractional linear ordinary differential equations.

Laplace Substitution Method for Solving Partial Differential Equations Involving Mixed Partial Derivatives

Summary

SujitHandibag, et.al, 2012, has proposed a new method, named Laplace substitution method (LSM), which was based on Laplace transform.

This new method with a convenient way to find exact solution with less computation as compared with Method of Separation of Variables(MSV) and Variation iteration method (VIM).

The proposed method solves linear partial differential equations involving mixed partial derivatives.

Findings

Laplace Substitution Method (LSM) is applicable to solve partial differential equations in which involves mixed partial derivatives and general linear term Ru(x, y) is zero.

The result of first two examples compared with (MSV) and (VIM), yields that these methods can be use alternatively for the solution of higher order initial value problem in which involves the mixed partial derivatives with general linear term Ru(x, y) is zero.

But the result of example number three yields that (LSM) is not applicable for those partial differential equations in which R u(x, y) 0.Consequently the (LSM) is promising and can be applied for other equations that appearing in various scientific fields.

Applications of Fractional Differential Equations

Summary

Mehdi Rahimy, 2010, has considered different definitions of

fractional derivatives,
Some kind of fractional differential equations were also studied
Some of their applications were also given.
Findings
First of all, most commonly used definitions of fractional derivative and there important
Properties were reviewed.
Then some examples about explicit solutions of some differential equations were discussed.
Some applications of fractional differential equations were stated viz:
1. Abel's integral equation
2. Viscoelasticity
3. Schrödinger equation
4. Analysis of Fractional Differential Equations
5. Food Science
6. Fractional Diffusion Equations
7. Fractional relaxation equation

IV. CONCLUSION

In recent decades, it has attracted interest of researches in several areas of science. Specially, in the field of physics applications of fractional calculus have gained considerable popularity. The application of Laplace transform is investigated to obtain an exact solution of some linear fractional differential equations. Solving some problems show that the Laplace transform is a powerful and efficient techniques for obtaining analytic solution of linear fractional differential equations.

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