THE SEQUENCE SPACE $l_m(\Delta, p, q, s)$ ON SEMI NORMED SPACES

K. Indrani[1], A.Sathyakala[2]
[1]Sr. Grade Lecturer, [2]Assistant professor
[1]Nirmala College for women, [2]A.S.L Pauls College of Engineering & Technology
Coimbatore, Anna University, Chennai, India.

***Abstract:* In this paper we define the sequence space $l_m(\Delta, p, q, s)$ on seminormed complex linear space, by using a sequence of orlicz functions. We study its some algebraic and topological properties. We give also some inclusion relations.**
***Keywords:* Orlicz function, Orlicz space, paranorm.**

1. INTRODUCTION

Let X be a complex linear space with zero element $\theta = (0,0,...)$ and $X = (X,q)$ be a seminormed space with the seminorm q. S(X) denotes the linear space of all sequence $x = (x_k)$ with $(x_k) \in X$ with the usual coordinate wise operations.

$\alpha x = (\alpha x_k)$ and $\mathrm{x} + \mathrm{y} = (x_k + y_k)$ for each $\alpha \in C$ where C denotes the set of complex numbers. If $\lambda = (\lambda_k)$ is a scalar sequence and $x \in s(x)$ then $\lambda \mathrm{x} = (\lambda_k \mathrm{x}_k)$.

Many authors including KINMAZ, CICDEM, A. BEKTAS defined new sequence spaces using a particular function like modulus function, orlicz function etc…

LINDERSTRAUSS and TZAFRIR have used an orlicz function to construct the sequence space,

$$l_M = \left\{ x \in \omega : \sum M\left(\frac{x_k}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

The spaces l_M becomes space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

Which is called an orlicz space. The space l_M is closely related to the space l_P which is an orlicz sequence space with $M(x) = x^p$ for $1 \le \mathrm{p} < \infty$.

Let **M** be a orlicz function, X be a seminormed space with seminorm $q, s \ge 0$ real number and let $p = p_k$ be a sequence of positive numbers .Then,

$$l_M(p,q,s) = \left\{ x \in S(X) \sum_{k=1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\mathrm{x}_k}{\rho} \right) \right) \right]^{\mathrm{p}_k} < \infty, \mathrm{s} \ge 0, \text{ for some } \rho > 0 \right\}$$

In this section the space $l_M(p,q,s)$ is extended to $l_M(\Delta, p, q, s)$ as follows.

$$l_M(\Delta, p,q,s) = \left\{ x \in S(X) \sum_{k=1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{x}_k)}{\rho} \right) \right) \right]^{\mathrm{p}_k} < \infty, \mathrm{s} \ge 0, \text{ for some } \rho > 0 \right\}$$

THEOREMS OF $l_M(\Delta, p, q, s)$:

Theorem: 1.1

Let $H = \sup_k p_k$, then $l_M(\Delta, p, q, s)$ is a linear space over the field C of complex numbers.

Proof:

Let $x, y \in l_M(\Delta, p, q, s)$ and $\alpha, \beta \in C$

In order to prove we need to find some ρ_3 such that

$$\sum_{k=1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\alpha \mathrm{x}_k + \beta \mathrm{y}_k)}{\rho_3} \right) \right) \right]^{\mathrm{p}_k} < \infty$$

Since $x, y \in l_M(\Delta, p, q, s)$

There exists positive numbers ρ_1, ρ_2 such that

$$\sum_{k=1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{x}_k)}{\rho_1} \right) \right) \right]^{\mathrm{p}_k} < \infty$$

$$\sum_{k=1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{y}_k)}{\rho_2} \right) \right) \right]^{\mathrm{p}_k} < \infty$$

Define $\rho_3 = \max\left(2|\alpha|\rho_1, 2|\beta|\rho_2\right)$

$$\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(\alpha x_k+\beta y_k)}{\rho_3}\right)\right)\right]^{p_k} \le$$

$$\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(\alpha x_k)}{\rho_3}\right)+q\left(\frac{\Delta(\beta y_k)}{\rho_3}\right)\right)\right]^{p_k}$$

$$\le \sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(\alpha x_k)}{2|\alpha|\rho_1}\right)+q\left(\frac{\Delta(\beta y_k)}{2|\beta|\rho_2}\right)\right)\right]^{p_k}$$

$$\le \sum_{k=1}^{\infty} \frac{1}{2^{p_k}} K^{-s}\left[M\left(q\left(\frac{\Delta x_k}{\rho_1}\right)+q\left(\frac{\Delta y_k}{\rho_2}\right)\right)\right]^{p_k}$$

$$\le D\sum_{k=1}^{\infty} \frac{1}{2^{p_k}} K^{-s}\left[M\left(q\left(\frac{\Delta x_k}{\rho_1}\right)\right)\right]^{p_k} +$$

$$D\sum_{k=1}^{\infty} \frac{1}{2^{p_k}} K^{-s}\left[M\left(q\left(\frac{\Delta y_k}{\rho_2}\right)\right)\right]^{p_k}$$

$$< \infty$$

Hence $\alpha x+\beta y \in l_M(\Delta,p,q,s)$ if x and y belongs to $l_M(\Delta,p,q,s)$ and $\alpha,\beta \in C$.

Where $D=\max(1,2^{H-1})$

Hence $l_M(\Delta,p,q,s)$ is a linear space.

Theorem: 1.2

$l_M(\Delta,p,q,s)$ is a paranormed (need not total paranorm) space with

$$g_\Delta(x)=\inf\left\{\rho^{pn/H} : \left[\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta x_k}{p}\right)\right)\right]^{p_k}\right]^{1/H} \le 1,\ n=1,2,3..\right\}$$

Where $H=\max\left(1,\sup_k p_k\right)$

Proof :

To prove that $g(x)=g(-x)$

$$g_\Delta(-x)=\inf\left\{\rho^{pn/H} : \left[\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(-x_k)}{p}\right)\right)\right]^{p_k}\right]^{1/H} \le 1,\ n=1,2,3..\right\}$$

Since

$$\Delta(-x)=\Delta(-x_1,-x_2,\ldots)=(-x_1+x_2,\ldots)$$
$$=(x_2-x_1,x_3-x_2,\ldots)$$
$$\Delta(-x)=(-1)(x_1-x_2,x_2-x_3,\ldots)$$
$$g\Delta(-x_k)=g((-1)\Delta x_k)=g(\Delta x_k)$$

Hence,

$$g_\Delta(-x)=\inf\left\{\rho^{pn/H} : \left[\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(-x_k)}{\rho}\right)\right)\right]^{p_k}\right]^{1/H} \le 1,\ n=1,2,3..\right\}$$

$$= g_\Delta(x)$$

$$g_\Delta(-x_k) == g_\Delta(x_k)$$

Taking $\alpha=\beta=1$ in $(*)$, we may write

$$\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(\alpha x_k+\beta y_k)}{\rho_3}\right)\right)\right]^{p_k}$$

$$\le \sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho_1}\right)\right)+M\left(q\left(\frac{\Delta(y_k)}{\rho_2}\right)\right)\right]^{p_k}$$

And then by using minkowski's inequality we get, $g(x+y)\le g(x)+g(y)$

Since $q(\theta)=0$ and $M(0)=0$, we get $g_\Delta(0)=0$

Finally, to prove that scalar multiplication is continuous.

Let λ be any number, since

$$g_\Delta(\lambda x)=\inf\left\{\rho^{pn/H}\left[\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(\lambda x_k)}{\rho}\right)\right)\right]^{p_k}\right]^{1/H} \le 1,\ n=1,2,3..\right\}$$

We may write,

$$g_\Delta(\lambda x)=\inf\left\{(\lambda x)^{pn/H}\left[\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(\lambda x_k)}{r}\right)\right)\right]^{p_k}\right]^{1/H} \le 1,\ n=1,2,3..\right\}$$

Where $r=\rho/\lambda$ .Since $|\lambda|^{p_k} \le \max\left(1,|\lambda|^H\right)$,

Then $|\lambda|^{p_k/H} \le \max\left(1,|\lambda|^H\right)$.

Hence

$$g(\lambda x)=\left(\max(1,|\lambda|^H)\right)^{1/H}$$

International Journal For Technological Research In Engineering
Volume 1, Issue 7, March-2014 *ISSN (Online): 2347 - 4718*

$$\inf\left\{(r)^{pn/H} : \left[\sum_{k=1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{x_k})}{r}\right)\right)\right]^{\mathrm{p_k}}\right]^{1/H} \le 1, \quad \mathrm{n}=1,2,3..\right\}$$

Which converges to zero as g(x) converges to zero in $l_M(\Delta, p, q, s)$.

Now suppose that $\lambda_{\mathrm{n}} \to 0$ as $\mathrm{n} \to \infty$ and $\mathrm{x} \in \mathrm{l_M}(\Delta, p, q, s)$

$$\sum_{k=N+1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{x_k})}{\rho}\right)\right)\right]^{\mathrm{p_k}} < \frac{\varepsilon}{2}$$

For some $\rho > 0$ this implies that

$$\left(\sum_{k=N+1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{x_k})}{\rho}\right)\right)\right]^{\mathrm{p_k}}\right)^{1/H} \le \frac{\varepsilon}{2}$$

If $0 < |\lambda| < 1$ then convexity of M implies,

$$\sum_{k=N+1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{x_k})}{\rho}\right)\right)\right]^{\mathrm{p_k}}$$

$$\le \sum_{k=N+1}^{\infty} K^{-s}\left[|\lambda|\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{x_k})}{\rho}\right)\right)\right]^{\mathrm{p_k}}$$

$$< \left(\frac{\varepsilon}{2}\right)^{H} \qquad \text{-----} \ (1)$$

Since M is continuous where in $[0, \infty)$ then

$$f(t) = \sum_{k=1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{t\mathrm{x_k}}{\rho}\right)\right)\right]^{\mathrm{p_k}}$$

is continuous at zero .So, there is *If* $0 > \delta > 1$ such that

$|f(t)| = \frac{\varepsilon}{2}$ for $0 < t < \delta$.

Let K be such that $|\lambda| < \delta$ then n>k, we have

$$\left(\sum_{k=1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\lambda_n \mathrm{x_n}}{\rho}\right)\right)\right]^{\mathrm{p_k}}\right)^{1/H} \le \frac{\varepsilon}{2} \quad \text{------}(2)$$

$$\left(\sum_{k=1}^{\infty} K^{-s}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\lambda_n \mathrm{x_n}}{\rho}\right)\right)\right]^{\mathrm{p_k}}\right)^{1/H} \le \varepsilon \ \text{ for } \mathrm{n} > \mathrm{k}$$

Hence the proof.

Theorem : 1.3

Let M, M_1, M_2 be orlicz function which satisfy Δ_2 condition and let s, s_1, s_2 be non-negative real numbers.

- If $\mathrm{s} > 1$, then $l_{M_1}(\Delta, p, q, s) \subseteq l_{M \circ M_1}(\Delta, p, q, s)$
- $l_{M_1}(\Delta, p, q, s) \cap l_{M_2}(\Delta, p, q, s) \subseteq l_{M_1 + M_2}(\Delta, p, q, s)$
- If $\mathrm{s_1} \le \mathrm{s_2}$, then $l_M(\Delta, p, q, s_2)$

Theorem : 1.4

- Suppose that $0 < r_k \le t$ for each k. then $l_M(\Delta, r, q) \subseteq l_M(\Delta, t, q)$
- $l_M(\Delta, q) \subseteq l_M(\Delta, q, s)$
- $l_M(\Delta, p, q) \subseteq l_M(\Delta, p, q, s)$

Proof:

Let $\mathrm{x} \in \mathrm{l_M}(\Delta, r, q)$

1). To Prove that $\mathrm{x} \in \mathrm{l_M}(\Delta, t, q)$

Let Let $\mathrm{x} \in \mathrm{l_M}(\Delta, r, q)$ then there exist some $\rho > 0$ such that

$$\sum_{k=1}^{\infty}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{x_k})}{\rho}\right)\right)\right]^{\mathrm{r_k}} < \infty$$

This implies that $\mathrm{M}\left(\mathrm{q}\left(\frac{\mathrm{x_i}}{\rho}\right)\right) \le 1$ for sufficiently large values of i.

Since M is non-decreasing we get

$$\sum_{k=1}^{\infty}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{x_k})}{\rho}\right)\right)\right]^{\mathrm{t_k}}$$

$$\le \sum_{k=1}^{\infty}\left[\mathrm{M}\left(\mathrm{q}\left(\frac{\Delta(\mathrm{x_k})}{\rho}\right)\right)\right]^{\mathrm{r_k}}$$

$$< \infty$$

There fore $\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]^{t_k} < \infty$

Hence $x \in l_M(\Delta, t, q)$.

There fore $l_M(\Delta, r, q) \subseteq l_M(\Delta, t, q)$

2). **To Prove that** $l_M(\Delta, q) \subseteq l_M(\Delta, q, s)$

Let $x \in l_M(\Delta, q)$

To Prove that $x \in l_M(\Delta, q, s)$

Let $x \in l_M(\Delta, q)$ then there exist some $\rho > 0$ such that

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right] < \infty$$

This implies that $M\left(q\left(\frac{x_i}{\rho}\right)\right) \leq 1$ for sufficiently large values of i.

Since M is non-decreasing we get

$$\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]$$

$$\leq \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]$$

$$< \infty$$

There fore $\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right] < \infty$

Hence $x \in l_M(\Delta, q, s)$.

There fore $l_M(\Delta, q) \subseteq l_M(\Delta, q, s)$

3). To Prove that $l_M(\Delta, p, q) \subseteq l_M(\Delta, p, q, s)$

Let $x \in l_M(\Delta, p, q)$

To Prove that $x \in l_M(\Delta, p, q, s)$

Let $x \in l_M(\Delta, p, q)$ then there exist some $\rho > 0$ such that

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right] < \infty$$

This implies that $M\left(q\left(\frac{x_i}{\rho}\right)\right) \leq 1$ for sufficiently large values of i.

Since M is non-decreasing we get

$$\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]^{p_k}$$

$$\leq \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]^{p_k}$$

$$< \infty$$

There fore $\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right] < \infty$

Hence $x \in l_M(\Delta, p, q, s)$.

There fore $l_M(\Delta, p, q) \subseteq l_M(\Delta, p, q, s)$

Hence the proof.

Theorem: 1.5

1). If $0 < p_k \leq 1$ for each k, then

$$l_M(\Delta, p, q) \subseteq l_M(\Delta, q)$$

2). If $p_k > 1$ for each k, then

$$l_M(\Delta, q) \subseteq l_M(\Delta, p, q)$$

Proof :

Let $x \in l_M(\Delta, p, q)$

To prove that $x \in l_M(\Delta, q)$

Then there exist some $\rho > 0$ such that

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]^{p_k} < \infty$$

This implies that $M\left(q\left(\frac{x_i}{\rho}\right)\right) \leq 1$

for sufficiently large values of i.

Since M is non-decreasing we get

$$\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]$$

$$\leq \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]^{p_k}$$

$$< \infty$$

Therefore

$$\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right] < \infty$$

Hence $x \in l_M(\Delta, q)$

There fore $l_M(\Delta, p, q) \subseteq l_M(\Delta, q)$

3). To prove that $l_M(\Delta, q) \subseteq l_M(\Delta, p, q)$

Let $x \in l_M(\Delta, q)$

then there exist some $\rho > 0$ such that

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right] < \infty$$

This implies that $M\left(q\left(\frac{x_i}{\rho}\right)\right) \le 1$

for sufficiently large values of i.
Since M is non-decreasing we get

$$\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]^{p_k}$$

$$\le \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]$$

$< \infty$

Therefore

$$\sum_{k=1}^{\infty} K^{-s}\left[M\left(q\left(\frac{\Delta(x_k)}{\rho}\right)\right)\right]^{p_k} < \infty$$

Hence $x \in l_M(\Delta, p, q)$

There fore $l_M(\Delta, q) \subseteq l_M(\Delta, p, q)$

Hence the proof.

REFERENCES

[1] BASARIR, some new sequence spaces and related matrix transformations, 100 year University Press, 6(1990), 32-38.

[2] CIGDEM A.BEKTAS AND YAVUZ ALTIN, Indian j.of.pure.appl.Math. (2003), 23-32.

[3] ESI.A and M.Et, On kothe-toeplitz duals of generalized sequence space, Bull. Malaysian Math. Soc (Second series) 23(2000), 23-32.

[4] ET.M and R. COLAK, On generalized difference sequence spaces Soochow jour. math .21(4) (1995)

[5] I.J. MADDOX, Elements of functional analysis, Camb Univ. Press, 1970.

[6] I.J.MADDOX, Spaces of strongly sumable sequences, Quart. jour. Math. 2 (18) (1976), 345-355.

[7] H. KIZMAZ, On certain sequence spaces, Canad. Math. Bull. 24(1981), 169-176.

[8] LINDENSTRAUSS.J and TZAFRIRI. L, On orlicz sequence spaces, Israel J.Math.10 (1971).

[9] ET.M, on some difference sequence spaces Doga-Tr.J.of.Math, 17 (1993), 18-24.

[10] ET.M, H.ALTINOK AND YAVAZ ALTIN, On generalized difference sequence spaces Applied Mathematics and computation, 154(2004), 167-173.

