International Journal For Technical Research In Engineering
Volume 7, Issue 5, January-2020 *ISSN (Online) : 2347-4718*

Role of Fourier transforms in differential equations

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Abstract : *In this paper, we discuss Jaccobian independent varibale functions as well as Theorem related to independent variables..*

1. Intoduction

Transformation is one of the operator which converts a mathematical expression into different form by using this method to solve simple way problem. It is powerful tools in treating various problems. Fourier transforms and integrals are useful in B.V.P.arising in Science as well as field of engineering. Jaccobian and independent variable has important role in the field of fourier transform.

1. Definition :

Let $u_1, u_2, u_3, ...u_n$ be the functions of n variables $x_1, x_2, ..., x_m$ then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \cdots\cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \cdots\cdots & \frac{\partial u_2}{\partial x_n} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & \frac{\partial u_3}{\partial x_n} \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, u_3, ...u_n$ with respective to $x_1, x_2, ..., x_m$. Generally we can write

$$\frac{\partial(u_1, u_2, u_3 .... u_n)}{\partial(x_1, x_2, x_3 ..... x_n)} \ or \ J(u_1, u_2, ..., u_n)$$

2. Definition :

Let $V_1, V_2, V_3, \ldots\ldots\ldots\ldots V_m$ be the functions of m variables $x_1, x_2, \ldots\ldots\ldots\ldots, x_m$ then the determinant

$$\begin{vmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial V_1}{\partial x_2} & \frac{\partial V_1}{\partial x_3} \ldots\ldots & \frac{\partial V_1}{\partial x_m} \\ \frac{\partial V_2}{\partial x_1} & \frac{\partial V_2}{\partial x_2} & \frac{\partial V_2}{\partial x_3} \ldots\ldots & \frac{\partial V_2}{\partial x_m} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial V_m}{\partial x_1} & \frac{\partial V_m}{\partial x_2} & \frac{\partial V_m}{\partial x_3} & \frac{\partial V_m}{\partial x_m} \end{vmatrix}$$

is called the Jacobian of $V_1, V_2, V_3, \ldots\ldots\ldots\ldots V_m$ with respective to $x_1, x_2, \ldots\ldots\ldots\ldots, x_m$. Generally we can write

$$\frac{\partial(V_1, V_2, \ldots. V_m)}{\partial(x_1, x_2, \ldots.. x_m)} \ or \ J(V_1, V_2, \ldots\ldots V_m)$$

If *u,v,w* are functions of three independent variables w.r.t. *x,y,z* is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Theorem (3.1) : If $V_1, V_2, V_3, \ldots\ldots\ldots\ldots V_m$ are functions $w_1, w_2, \ldots\ldots\ldots\ldots, w_m$ and $w_1, w_2, \ldots\ldots\ldots\ldots, w_m$ are the functions of $x_1, x_2, \ldots\ldots\ldots\ldots, x_m$ then

$$\frac{\partial(V_1, V_2, \ldots., V_m)}{\partial(x_1, x_2, \ldots.., x_m)} = \frac{\partial(V_1, V_2, \ldots., V_m)}{\partial(w_1, w_2, \ldots.., w_m)} \cdot \frac{\partial(w_1, w_2, \ldots., w_m)}{\partial(x_1, x_2, \ldots.., x_m)}$$

Proof : Suppose that $V_1, V_2, V_3, \ldots\ldots\ldots\ldots V_m$ are functions of $w_1, w_2, \ldots\ldots\ldots\ldots, w_m$ and $w_1, w_2, \ldots\ldots\ldots\ldots, w_m$ are functions of $x_1, x_2, \ldots\ldots\ldots\ldots, x_m$ then

$$\frac{\partial V_1}{\partial x_1} = \frac{\partial V_1}{\partial w_1} \cdot \frac{\partial w_1}{\partial x_1} + \frac{\partial V_1}{\partial w_2} \cdot \frac{\partial w_2}{\partial x_1} + \ldots\ldots + \frac{\partial V_1}{\partial w_m} \cdot \frac{\partial w_m}{\partial x_1}$$

$$= \sum_{j=1}^{n} \frac{\partial V_1}{\partial w_j} \cdot \frac{\partial w_j}{\partial x_1}$$

$$= \frac{\partial V_1}{\partial x_1} = \frac{\partial V_1}{\partial w_1} \cdot \frac{\partial w_1}{\partial w_j} + \frac{\partial V_1}{\partial w_2} \cdot \frac{\partial w_2}{\partial x_2} + \ldots\ldots. + \frac{\partial V_1}{\partial w_m} \cdot \frac{\partial w_m}{\partial x_2}$$

$$= \sum_{j=1}^{n} \frac{\partial V_1}{\partial w_j} \cdot \frac{\partial w_j}{\partial x_2}$$

....

....

$$\frac{\partial(V_1, V_2, \ldots.., V_m)}{\partial(w_1, w_2 \ldots\ldots, w_m)} \times \frac{\partial(w_1, w_2, \ldots.., w_m)}{\partial(x_1, x_2 \ldots\ldots, x_m)}$$

$$= \begin{vmatrix} \frac{\partial V_1}{\partial w_1} & \frac{\partial V_1}{\partial w_2} & \ldots.. & \frac{\partial V_1}{\partial w_m} \\ \frac{\partial V_2}{\partial w_2} & \frac{\partial V_2}{\partial w_2} & \ldots.. & \frac{\partial V_2}{\partial w_1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial V_m}{\partial w_1} & \frac{\partial V_m}{\partial w_2} & \ldots. & \frac{\partial V_m}{\partial w_m} \end{vmatrix} \times \begin{vmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial x_2} & \ldots.. & \frac{\partial w_1}{\partial x_m} \\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial x_1} & \ldots.. & \frac{\partial w_2}{\partial x_1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial w_m}{\partial x_1} & \frac{\partial w_m}{\partial x_2} & \ldots. & \frac{\partial w_m}{\partial x_m} \end{vmatrix}$$

$$
=\begin{vmatrix}
\sum \frac{\partial V_1}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_1} & \sum \frac{\partial V_1}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_2} & ..... & \sum \frac{\partial u_1}{\partial y_j}\cdot\frac{\partial w_j}{\partial x_m} \\
\sum \frac{\partial V_2}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_1} & \sum \frac{\partial V_2}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_2} & ..... & \sum \frac{\partial u_2}{\partial y_j}\cdot\frac{\partial w_j}{\partial x_m} \\
. & . & . & . \\
. & . & . & . \\
. & . & . & . \\
. & . & . & . \\
\sum \frac{\partial V_m}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_1} & \sum \frac{\partial V_m}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_2} & .... & \sum \frac{\partial V_m}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_m}
\end{vmatrix}
$$

$$
=\begin{vmatrix}
\frac{\partial V_1}{\partial x_1} & \frac{\partial V_1}{\partial x_2} & ..... & \frac{\partial V_1}{\partial x_m} \\
\frac{\partial V_2}{\partial x_2} & \frac{\partial V_2}{\partial x_2} & ..... & \frac{\partial V_2}{\partial x_1} \\
. & . & . & . \\
. & . & . & . \\
. & . & . & . \\
. & . & . & . \\
\frac{\partial V_m}{\partial x_1} & \frac{\partial V_m}{\partial x_2} & .... & \frac{\partial V_m}{\partial x_m}
\end{vmatrix} = \frac{\partial(V_1,V_2,.....,V_n)}{\partial(x_1,x_2,.....,x_n)}
$$

Theorem 3.2: If $V_1,V_2,V_3,............V_m$ and $y_1,y_2,............,y_m$ are implicity connected by m equations as :

$$
\begin{array}{ll}
f_1(V_1,V_2,V_3,............V_m,y_1,y_2,............,y_m)=0 \\
f_2(V_1,V_2,V_3,............V_m,y_1,y_2,............,y_m)=0 \\
. & . \\
. & . \\
. & . \\
f_n(V_1,V_2,V_3,............V_m,y_1,y_2,............,y_m)=0
\end{array}
$$

$$\frac{\partial(f_1, f_2, \ldots\ldots, f_n)}{\partial(u_1, u_2 \ldots\ldots, u_n)} \cdot \frac{\partial(V_1, V_2, \ldots\ldots, V_n)}{\partial(x_1, x_2 \ldots\ldots, x_n)}$$

$$= (-1)^n \frac{\partial(f_1, f_2, \ldots\ldots, f_n)}{\partial(x_1, x_2 \ldots\ldots, x_n)}$$

Proof : Differentiating the above the given releations with respect to $x_1, x_2, \ldots\ldots\ldots\ldots, x_m$ we set

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial V_1} \cdot \frac{\partial V_1}{\partial x_1} + \frac{\partial f_1}{\partial V_2} \cdot \frac{\partial V_2}{\partial x_1} + \ldots + \frac{\partial f_1}{\partial V_n} \cdot \frac{\partial V_n}{\partial x_1} = 0$$

$$\frac{\partial f_1}{\partial x_1} + \sum \frac{\partial f_1}{\partial w_j} \cdot \frac{\partial w_j}{\partial x_1} = 0$$

$$\sum \frac{\partial f_1}{\partial w_j} \cdot \frac{\partial w_j}{\partial x_j} = \frac{-\partial f_1}{\partial x_1}$$

$$\sum \frac{\partial f_1}{\partial w_j} \cdot \frac{\partial w_j}{\partial x_2} = \frac{-\partial f_1}{\partial x_2}$$

$$\text{and } \sum \frac{\partial f_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_n} = \frac{-\partial f_1}{\partial x_n}$$

Now $$\frac{\partial(f_1, f_2, \ldots\ldots, f_n)}{\partial(V_1, V_2 \ldots\ldots, V_m)} \times \frac{\partial(w_1, w_2, \ldots\ldots, w_n)}{\partial(x_1, x_2 \ldots\ldots, x_n)}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial V_1} & \frac{\partial f_1}{\partial V_2} & \ldots\ldots & \frac{\partial f_1}{\partial V_n} \\ \frac{\partial f_2}{\partial V_2} & \frac{\partial f_2}{\partial V_2} & \ldots\ldots & \frac{\partial f_2}{\partial V_1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial f_n}{\partial V_1} & \frac{\partial f_n}{\partial V_2} & \ldots\ldots & \frac{\partial f_n}{\partial V_n} \end{vmatrix} \times \begin{vmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial V_1}{\partial x_2} & \ldots\ldots & \frac{\partial V_1}{\partial x_n} \\ \frac{\partial V_2}{\partial x_1} & \frac{\partial V_2}{\partial x_1} & \ldots\ldots & \frac{\partial V_2}{\partial x_N} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial V_n}{\partial x_1} & \frac{\partial V_n}{\partial x_2} & \ldots\ldots & \frac{\partial V_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \sum \frac{\partial f_1}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_1} & \sum \frac{\partial f_1}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_2} & ..... & \sum \frac{\partial f_1}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_n} \\ \sum \frac{\partial f_2}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_1} & \sum \frac{\partial f_2}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_2} & ..... & \sum \frac{\partial f_2}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \sum \frac{\partial f_n}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_1} & \sum \frac{\partial f_n}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_2} & .... & \sum \frac{\partial f_n}{\partial w_j}\cdot\frac{\partial w_j}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-\partial f_1}{\partial x_1} & \frac{-\partial f_1}{\partial x_2} & ..... & \frac{-\partial f_1}{\partial x_n} \\ \frac{-\partial f_2}{\partial x_1} & \frac{-\partial f_2}{\partial x_2} & ..... & \frac{-\partial f_2}{\partial x_n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{-\partial f_n}{\partial x_1} & \frac{-\partial f_n}{\partial x_2} & .... & \frac{-\partial f_n}{\partial x_n} \end{vmatrix} = (-1)^n \frac{\partial(f_1, f_2, ..., f_n)}{\partial(x_1, x_2, ..., x_n)}$$

Theorem 3.3: If $V_1, V_2, V_3, ...V_m$ be functions of independent variables $x_1, x_2, ..., x_n$. If $f(V_1, V_2, V_3, ...V_m) = 0$ if and only if

$$\frac{\partial(V_1, V_2, V_3, ...V_m)}{\partial(x_1, x_2, x_3, ...x_m)} = 0$$

Proof : Suppose that $f(V_1, V_2, V_3, ...V_m) = 0$ ....(1)

Differentiating (1) w.r.t. variable $x_1, x_2, ..., x_m$ then

$$\frac{\partial f}{\partial V_1}\cdot\frac{\partial V_1}{\partial x_1} + \frac{\partial f}{\partial V_2}\cdot\frac{\partial V_2}{\partial x_1} + ... + \frac{\partial f}{\partial V_n}\cdot\frac{\partial u_n}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial V_1}\cdot\frac{\partial V_1}{\partial x_1}+\frac{\partial f}{\partial V_2}\cdot\frac{\partial V_2}{\partial x_1}+\ldots+\frac{\partial f}{\partial V_n}\cdot\frac{\partial u_n}{\partial x_1}=0$$

$$\vdots$$

$$\frac{\partial f}{\partial V_1}\cdot\frac{\partial V_1}{\partial x_n}+\frac{\partial f}{\partial V_2}\cdot\frac{\partial V_2}{\partial x_n}+\ldots+\frac{\partial f}{\partial V_n}\cdot\frac{\partial V_n}{\partial x_n}=0$$

Now to eliminate $\frac{\partial f}{\partial V_1},\frac{\partial f}{\partial V_2},..........\frac{\partial f}{\partial V_n}$ then

$$=\begin{vmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial V_2}{\partial x_2} & \ldots.. & \frac{\partial V_n}{\partial x_n} \\ \frac{\partial V_1}{\partial x_1} & \frac{\partial V_2}{\partial x_2} & \ldots.. & \frac{\partial V_n}{\partial x_n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial V_1}{\partial x_n} & \frac{\partial V_2}{\partial x_n} & \ldots. & \frac{\partial V_n}{\partial x_n} \end{vmatrix}=0$$

i.e. $\frac{\partial(V_1,V_2,V_3,............V_m)}{\partial(x_1,x_2,x_3,............x_m)}=0$

converely, suppose that $\frac{\partial(V_1,V_2,V_3,............V_m)}{\partial(x_1,x_2,x_3,............x_m)}=0$

then we have to prove that $f(V_1,V_2,V_3,............V_m)=0$. The equation connecting $V_1,V_2,V_3,............V_m$ and $x_1,x_2,............,x_n$ are always capable, by eliminating of being transformed into the following form

$$F_1(x_1, x_2, \ldots\ldots x_m, V_1) = 0$$
$$F_2(x_2, x_3, \ldots\ldots x_m, V_2) = 0$$
$$. \qquad\qquad\qquad\qquad .$$
$$. \qquad\qquad\qquad\qquad .$$
$$F_n(x_m, u_1, u_2, \ldots\ldots u_m) = 0$$

$$\therefore J(V_1, V_2, \ldots\ldots, V_m) = (-1)^n \frac{\dfrac{\partial F_1}{\partial x_1}, \dfrac{\partial F_2}{\partial x_2}, \ldots\ldots\ldots\ldots \dfrac{\partial F_n}{\partial x_n}}{\dfrac{\partial F_1}{\partial V_1}, \dfrac{\partial F_2}{\partial V_2}, \ldots\ldots\ldots\ldots \dfrac{\partial F_n}{\partial V_n}}$$

But $J(V_1, V_2, V_3, \ldots\ldots\ldots\ldots V_m) = 0$

$$\frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \ldots \frac{\partial F_n}{\partial V_n} = 0$$

$\therefore \dfrac{\partial F_j}{\partial x_j} = 0$ for same value of $j, \quad 1 \le j \le m$

$$\therefore F_j(x_{j+1}, x_{j+2}, \ldots x_n, V_1, V_2 \ldots V_j) = 0$$

i.e. $F_{j+1} = 0, \ldots\ldots, F_n = 0$

4. Transformation of $\Delta^2 V$

We know that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \Delta^2 V \tag{4.1}$$

If $x = r\cos\theta, y = r\sin\theta,$ then we know

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} \tag{4.2}$$

The operator V^2 stands for $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

The transformation formulae are as follows:-

$x = r\,\sin\theta\cos\phi$ If $r\sin\theta = u,$

$y = r\,\sin\theta\cos\phi$ then $x = u\cos\phi,$

$z = r\,\cos\theta$ $y = u\sin\phi.$

By (4.2)

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial u^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2} \tag{4.3}$$

Where $z = r\,\cos\theta$
$u = r\sin\theta,$

$$\therefore \quad \frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} \tag{4.4}$$

Additing (4.3) and (4.4), we get

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial u^2}=\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} \tag{4.5}$$

We know that

$$\frac{\partial V}{\partial u}=\frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial u}+\frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial u}$$

$z = r\,\cos\theta$ $u = r\sin\theta,$

$$\therefore \quad r=\sqrt{(z^2+u^2)}, \qquad \frac{\partial r}{\partial u}=\frac{u}{\sqrt{(z^2+u^2)}}=\sin\theta$$

$$\theta=\tan^{-1}\frac{u}{z}, \qquad \therefore \frac{\partial \theta}{\partial u}=\frac{z}{u^2+z^2}=\frac{\cos\theta}{r}$$

$$\frac{\partial V}{\partial u}=\sin\theta\frac{\partial V}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial V}{\partial \theta}$$

i.e. $$\frac{1}{u}\left(\frac{\partial V}{\partial u}\right)=\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\cot\theta\frac{\partial V}{\partial \theta}$$

Substituting this value in (4.5), we get

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\cot\theta\frac{\partial V}{\partial \theta}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$

$$=\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{r^2}\cot\theta\frac{\partial V}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$\therefore \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{r^2}\cot\theta\frac{\partial V}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Example

1) If $u^3+v^3=x+y$, $u^2+v^2=x^3+y^3$, show that $\frac{\partial(u,v)}{\partial(x,y)}=\frac{1}{2}\quad\frac{y^2-x^2}{uv(u-v)}$

Sol^n : Given

$$u^3+v^3=x+y \text{ and } u^2+v^2=x^3+y^3$$

Suppose $\quad f_1=u^3+v^3-x-y=0$

$$f_2=u^2+v^2-x^3-y^3=0$$

Now $\frac{\partial(u,v)}{\partial(x,y)}=(-1)^2\quad\frac{\partial(f_1,f_2)}{\partial(x,y)}\div\frac{\partial(f_1,f_2)}{\partial(u,v)}$

$$=\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{vmatrix}\div\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix}$$

$$\begin{vmatrix}-1 & -1\\ -3x^2 & -3y^2\end{vmatrix}\div\begin{vmatrix}3u^2 & 3v^2\\ 2u & 2v\end{vmatrix}=\frac{(3y^2-3x^2)}{6(u^2v-4v^2)}$$

$$=\frac{1}{2}\frac{(y^2-x^2)}{uv(u-v)}$$

$$=\frac{1}{2}[F(\lambda+a)+F(\lambda-a)]$$

2) If $u=\frac{x+y}{z}, v=\frac{y+z}{x}, w=\frac{y(x+y+z)}{xz}$ Show that *u,v,w* are not indepndent and find the relation between them.

Sol^n : Given $u=\frac{x+y}{z}, v=\frac{y+z}{x}, w=\frac{y(x+y+z)}{xz}$

$$J(u,v,w)=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$=\begin{vmatrix} \frac{1}{z} & \frac{1}{z} & \frac{(x+y)}{z^2} \\ \frac{-(y+z)}{x^2} & \frac{1}{x} & \frac{1}{x} \\ \frac{-(y^2-yz)}{x^2 z} & \frac{x+2y+z}{xz} & \frac{-xy-y^2}{xz^2} \end{vmatrix}$$

Taking $\frac{1}{z^2},\frac{1}{x^2},\frac{1}{x^2y^2}$ common from R_1,R_2,R_3 then

$$J(u,v,w)=\frac{1}{x^4z^4}=\begin{vmatrix} z & z & -(x+y) \\ -(y+z) & x & x \\ -(y^2z+z^2y) & x^2z+2xyz+xz^2 & -(x^2y+xy^2) \end{vmatrix}$$

$C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1$ then

$$J(u,v,w)=\frac{1}{x^4z^4}\begin{vmatrix} z & z & -(x+y+z) \\ -(y+z) & (x+y+z) & (x+y+z) \\ -yz(y+z) & z(x^2+y^2+2xy+z^2(x+y) & -xy(x+y)+yz(x+z) \end{vmatrix}$$

$$=\frac{1}{x^4z^4}\begin{vmatrix} z & 0 & -(x+y+z) \\ -(y+z) & (x+y+z) & (x+y+z) \\ -yz(y+z) & z(x+y)+(z+x+y) & (x+y+z)+(yz-xy) \end{vmatrix}$$

$$=\frac{(x+y+z)^2}{x^4z^4}\begin{vmatrix} z & 0 & -1 \\ -(y+z) & 1 & 1 \\ -yz(y+z) & z(x+y) & (yz-xy) \end{vmatrix}$$

$$=\frac{(x+y+z)^2}{x^4z^4}\begin{vmatrix} z & 0 & -1 \\ -y & 1 & 1 \\ -y^2+yz^2+yz^2-xyz & z(x+y) & (yz-xy) \end{vmatrix}$$

$$=\frac{(x+y+z)^2}{x^4z^4}(-1)\begin{vmatrix} -y & 1 \\ -yz(x+y) & z(+y) \end{vmatrix}$$

$$=\frac{(x+y+z)^2}{x^4z^4}(-1)\left[-yz(x+y)+yz(x+y)\right]=0$$

Since Jacobian function J=0. Therefore the given functions are not independent.

Now, $uv=\dfrac{xy+y^2+yz+zx}{zx}=\dfrac{y(x+y+z)}{xz}+1=w+1$

$\therefore uv=w+1$ is required relation between them.

3) If u is a function of r alone, when $r^2=x^2-y^2$ show that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}$

Sol^n :
$$r^2=x^2-y^2 \tag{1}$$

$$2r\frac{dr}{dx}=2x \Rightarrow \frac{dr}{dx}=\frac{x}{r}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}=\frac{x}{r}\frac{\partial u}{\partial r} \tag{2}$$

$$\frac{\partial}{\partial x}=\frac{x}{r}\frac{\partial}{\partial r}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{x}{r}\frac{\partial}{\partial r}\left(\frac{x}{r}\frac{\partial u}{\partial r}\right)$$

$$=\frac{1}{r}\frac{\partial u}{\partial x}+\frac{x^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right)$$

$$=\frac{1}{r}\frac{\partial u}{\partial r}-\frac{x^2}{r^3}\frac{\partial u}{\partial r}+\frac{x^2}{r^2}\frac{\partial^2 u}{dr^2} \qquad (3)$$

Similarly $$\frac{\partial^2 u}{\partial y^2}=-\left[\frac{1}{r}\frac{\partial u}{\partial x}-\frac{y^2}{r^3}\frac{\partial u}{\partial r}+\frac{y^2}{x^2}\frac{\partial^2 u}{\partial r^2}\right]$$

$$=-\frac{1}{r}\frac{\partial u}{\partial r}+\frac{y^2}{r^3}\frac{\partial u}{\partial r}-\frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} \qquad (4)$$

Taking addition of (3) and (4) then

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{-1}{r^3}\frac{\partial u}{\partial r}\left(x^2-y^2\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial r^2}\left(x^2-y^2\right)$$

$$=-\frac{1}{r^2}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}$$ which is required solution.

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