STABILITY

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Abstract : In the present we discuss the study of stability and Basic theorem and uniqueness and stability theorem related to Lipschitz local conditions.

1. INTRODUCTION:

The stability of solution to difference equations of order k is

$$y_{n+1} = F(n, y_{n+1}.........y_{n-k+t}) \qquad (1.1)$$

The function $F : I \times R^k \to R^+$

For unique solution to an equation of the form (1) with initial values

$$y_0, y_1, ...y_{k-1}$$

The above equation (1) can be written in form of vector

$$x_{n+1} = F(n, x_n) \qquad (1.2)$$

Where

$$x_n = (y_n, y_{n+1}.........y_{n-k+t})$$

We can write equation (2) in the linear form $f(n, x_n) = C(n)x_n + d(n)$

If d(n) becomes zero then it is called homogeneous and $x_n = 0$ then homogeneous linear equation is an equilibrium solution . Summation difference equation in the form of can be written as

$$y_{n+1} = F(n, y_n, y_{n+1}.........y_{n-k+t}) \qquad (1.3)$$

Agarwal [1], kelley and petorson [2], S.Elaydi, F.M.Danam [3] had developed theory of difference equation and stability some comparison theorem on difference equation and summation equation related to stability obtained by K.L.Bondar [4 to 6] V. Lakshmikantham, D. Trigiante pundir and pundir [7,8] so give motivate work in stability.

2. DIFINITIONS OF STABILITY AND THEIR TYPES:

Definition 2.1: Consider the difference equation

$$x_{n+1} = f(n, x_n)$$

Let $\{x_n\}$ and $\{t_n\}$ be the solution with initial condition $x_0 = \alpha \in R^K$ and $t_0 = \beta \in R^k$ The $\{x_n\}$ is said to be stability if given $\epsilon > 0$ then there exists $\delta > 0$ such that

$$|x_n - t_n| < \in \text{ whenever } |\alpha - \beta| < \delta$$

Every solution of (2) is bounded then it is stable.

Lakshmikantham and Trigiante [7] gives the idea of definition of stability in difference equation in the form

$$x_{n+1} = C(n)x_n + h(n) \qquad (2.1)$$

Definition 2.2 : The solution $\{x_n\}$ of (2.1) is stable if and only if $|x_n - t_n|$ is bounded for any other solution t_n of (2.1)

Definition 2.3: Consider the difference equation $x_{n+1} = F(n, x_n)$ Let $\{x_n\}$ and $\{t_n\}$ be the solutions with initial conditions $x_0 = \alpha$ and

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$t_0 = \beta$ then the solution $\{x_n\}$ is said to be asymptotically stable if given $\in = 0$ and then there exist $\delta > 0$ such that $|x_n - t_n| \to 0$ whenever $|\alpha - \beta| < \delta.$

It would be noted that Napoles Valdes [15] dealt with ordinary summation-difference equation of second order:

Definition 2.4: Consider the difference equation

$$x_{n+1} = F(n, x_n)$$

Suppose that $\{x_n\}$ and $\{t_n\}$ be the solutions with initial conditions $x_0 = \alpha$ and $t_0 = \beta$ then the solution $\{x_n\}$ is said to be asymptotically stable if there exists constants a, $\delta > 0$ and $\gamma \in (0,1)$ such that $|x_n - t_n| \le a|\alpha - \beta| r^n$ Whenever $|\alpha - \beta| \le \delta.$

Theorem 2.1: Let $\{x_n\}$ be the linear autonomous equation $x_{n+1} = Ax_n + b$ -(2.1.1) is exponentially stable if and only if it is asymptotically stable.

Definition 2.5: Consider a system of equation in first order in the form of

$$\Delta x = f(x, y)$$
$$\Delta y = f(x, y)$$

is said to be autonomous when its independent variable t does not appear explicitly.

3. HOMOGENEOUS LINEAR DIFFERENCE EQUATION.

Consider the homogeneous linear difference test Equation.

$$\Delta y(t) = \lambda y(t), \quad y(0) = y_0 \quad (3.1)$$

We can easily show that all the solution to (3.1) are stable with respective to small perturbation in the initial value condition y_0 if $\mathrm{Re}\, l\lambda \le 0$ and asymptotically stable if $\mathrm{Re}\, l\lambda < 0$

If $\mathrm{Re}\,\lambda \le 0$ then all solution to (3.1) are bounded. On the other hand $\mathrm{Re}\,\lambda \ge 0$ the solution for all h > 0

If Eq^n (3.1) is approximated using a simple numerical scheme, one obtains a linear coefficient difference equation of fixed finite order r. The equation can be expressed in the matrix –vector form

$$y_n = Ay_{n-1}$$

Where $y_n = (x_n, x_{n-1}, \ldots\ldots x_{n-(r-1)})^T$ and the matrix A contains the difference equation in the first row and shift operators in the rows 2,3,…………r. The dynamical behavior of solutions to (3.2) is determined by the Eigen values of A.

The situation is simplest whenever the Eigen values, λi of A are distinct all solutions are asymptotically stable when $Max|\lambda i| < 1$ stable. When $Max|\lambda i| = 1.$ and unstable when $Max|\lambda i| > 1$

In cases where a repeated Eigen value of magnitude 1 is the largest Eigen value and the corresponding Jordan block is not diagonal all solutions are unstable.

Remark :

1. If all $|\lambda i|<1$ then all solution are bounded and asymptotically stable
2. If all $|\lambda i|\leq 1$ and only simple roots (Multiplicity one) satisfy $|\lambda i|=1$, then all solutions are bounded and stable.
3. If all λi are simply and there are K(<r) such roots satisfying $|\lambda i|\leq 1$ then there is K-dimensional set of initial values leading to bounded solutions, all of which are unstable.

In other words stability corresponds to bounded ness of all solutions and instability a rises as soon as any other unbounded solutions exists. In case where all λi satisfy $|\lambda i|>1$ there will be exists a unique bounded solution y=0 corresponding to zero initial value.

The conventional approach to the stability analysis of difference equations of type we discuss is to being with detailed analysis of autonomous linear equations of the form.

$$y_{n+1} = Ay_n + b \qquad (3.3)$$

The analysis of non-autonomous linear equation of the form

$$y_{n+1} = A(n)y_n + b(n) \qquad (3.4)$$

To investigate the stability and bounded ness of solution to nonlinear equations, the usual approach is to concentrate on equations that are related in some way to the linear equations where behavior is known to be covered by existing theory.

Consider the equations of the form

$$y_{n+1} = Ay_n + f(y_n) \qquad (3.5)$$

Or

$$y_{n+1} = A(n)y_n + f(n, y_n) \qquad (3.6)$$

The numerical solutions to nonlinear differential equations of the form

$$\Delta y(t) = f(t, y(t)) \qquad (3.7)$$

This is the basic idea behind the use of linear test equations to analysis the stability of numerical methods.

4. A UNIFIED STABILITY THEORY:

In this section, we provide a unified theory for the analysis of stability and bounded ness of solution of autonomous and non-autonomous difference equations, both linear and nonlinear. Our aim in the analysis is the Lipschitz condition, familiar from several other area of mathematics. In this situation the use of Lipschitz condition enables us to develop a direct theory that applies both to linear and non-linear problems.

The concept of Lipschitz stability for difference equations was introduced in [3, 4]. Its application to difference equations is mentioned the book by Agarwal [1] published in 1992 provides an excellent survey of the state of the art at that time and focuses on other approaches to stability analysis.

Definition (4.1) (Uniform Lipschitz condition). Let *f*(*x*,*y*) be a function defined for $y \in Y, .x \in X$ where X is some arbitrary set and Y is a normed space then f satisfies a uniform Lipschitz condition on $X \times Y$ with respective to its second and argument if there exists a constant L such that

$$\| f(x,y) - f(x,z) \| \le L \| y - z \| \quad (4.1)$$

for every $x \in X$ and for all $y, z \in y$

Definition (4.2) (Local Lipschitz condition) : Let *f*(*x*,*y*) be a function defined on $x \in X, y \in Y$ then f satisfies a local Lipschitz condition with respective to its second argument in a neighborhood D of some point w if there exists a constant L^D such that

$$\| f(x,y) - f(x,z) \| \le L^D \| y - z \| \quad (4.2)$$

for every $x \in X$ and all $y, z \in D.$

We consider the general difference equation of the term

$$y_{n+1} = f(n, y_n). \qquad (4.3)$$

The sequence $\{y_n\}$ consists of K-vector and function f may be is general. The function f varies according to the value of its first argument. Eqn. (4.3) is called non autonomous equation otherwise autonomous. It is usually regarded as an IVP and the sequence $\{y_n\}$ generated is easily shown to be unique given a starting value y_0.

Let ∅ be any solution eq^n with initial value y_0 is

$$y_n = \phi(n, y_0)$$

BASIC THEOREM :

THEOREM :

4.1 The sequence $\{y_n\}$ satisfy an autonomous difference equation of the form

$$y_{n+1} = f(y_n) \qquad (4.4)$$

When the function f satisfies a uniform Lipschitz condition with Lipschitz constant L<1 then every solution is asymptotically stable. If f satisfies a uniform Lipschitz condition with Lipschitz constant L=1 then every solution to (4.5) is stable. Further either all solutions are bounded or all solution are unbounded.

4.2 BASIC THEOREM FOR NONAUTONOMOUS PROBLEM

Let $\{y_n\}$ be sequence satisfy a difference equation of the form

$$y_{n+1} = f(n, y_n). \qquad (4.5)$$

Where the function f satisfies for each value n, a uniform Lipschitz condition with respective to its second argument with Lipschitz constant $Ln \le M < 1$ then every solution of (4.5) is asymptotically stable. Further all solutions are bounded or all solutions are unbounded.

THEOREM:

4.3 LOCAL LIPSCHITZ UNIQUENESS AND STABILITY THEOREM

Let $\{y_n\}$ be sequence satisfy a difference equation of the form

$$y_{n+1} = f(n, y_n) \qquad (4.6)$$

and d be an equilibrium solution of (4.6) (in other words,) we assume that $f(n,d) = d$ for all $n \in N$. Assume that for some sphere S whose center is d the function f satisfies.

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