Properties of Fourier Transforms in Engineering field

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Abstract : *In this paper, we discuss some properties of integral transforms, modulation theorem and sine and cosine integral transform also some problems are mentioned.*

1. Intoduction

Generally, integral transforms has many applications in the field of ordinary and partial differential equations. By the help of this transform to solve the differential equations which depends upon initial and boundary conditions. Some properties in integral transform has important role to solve problem in simple way.

2. PROPERTIES AND THEOREMS OF FOURIER TRANSFORMS

Linearity Property 2.1: If $F(\lambda)$ and $G(\lambda)$ are Fourier transform of $f(x)$ and $g(x)$ respectively, and if we use notation $F[f(x)] = F(\lambda)$ *and* $F[g(x)] = G(\lambda)$, then

$$F[c_1 f(x) + c_2 g(x)] = c_1 F(\lambda) + c_2 G(\lambda)$$

where c_1 *and* c_2 are constants

Proof : By definition of Fourier transform, we have

$$F(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx \;\; and \;\; G(\lambda) = \int_{-\infty}^{\infty} g(x) e^{-i\lambda x} dx$$

$$\therefore \qquad F[c_1 f(x) + c_2 g(x)] = \int_{-\infty}^{\infty} [c_1 f(x) + c_2 g(x)] e^{-i\lambda x} dx$$

$$= c_1 \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx + c_2 \int_{-\infty}^{\infty} e^{-i\lambda x} dx$$

$$= c_1 F(\lambda) + c_2 G(\lambda)$$

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Change of Scale Property 2.2 : If $F(\lambda)$ is the complex Fourier transform of $f(x)$,

$$F[f(ax) = \frac{1}{a}F\left(\frac{\lambda}{a}\right)$$

Proof : We have, $F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx$

$$\therefore \quad F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx, \text{ and put } ax = 1$$

$$= \int_{-\infty}^{\infty} f(t)e^{-i\lambda x}\frac{dt}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f(t)e^{-i\lambda x}dt$$

$$= \frac{1}{a}F\left(\frac{\lambda}{a}\right)$$

Shifting Property 2.3 : If $F(\lambda)$ is the complex Fourier transform of $g(x)$, then

$$F[g(x-a)] = e^{-i\lambda a}F(\lambda)$$

Proof : We have $F[g(x)] = \int_{-\infty}^{\infty} g(x)e^{-i\lambda x}dx$

$$\therefore \quad F[g(x-a)] = \int_{-\infty}^{\infty} g(x-a)e^{-i\lambda x}dx, \text{ and put } x = t + a$$

$$= \int_{-\infty}^{\infty} g(t)e^{-i\lambda(t+a)}dt = e^{-i\lambda x}\int_{-\infty}^{\infty} g(t)e^{-i\lambda x}dt$$

$$= e^{-i\lambda a}F(\lambda).$$

Modulation Theorem 2.4: If $F(\lambda)$ is the complex Fourier transform of $g(x)$, then

$$F[g(x)] = \frac{1}{2}[F(\lambda + a) + F(\lambda - a)]$$

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Proof : We have $F[f(x)] = \int_{-\infty}^{\infty} g(x) e^{-i\lambda x} dx$

$$\therefore \qquad F[g(x)\cos ax] = \int_{-\infty}^{\infty} f(x)\cos axe^{-i\lambda x} dx, = \int_{-\infty}^{\infty} g(x)\left(\frac{e^{iax} + e^{-iax}}{2}\right) e^{-i\lambda x} dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} g(t) e^{-i(\lambda+a)x} dt + \int_{-\infty}^{\infty} g(x) e^{-i(\lambda-a)x} dx\right]$$

$$= \frac{1}{2}[F(\lambda + a) + F(\lambda - a)]$$

Note : If $F_s(\lambda)$ & $F_c(\lambda)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then

(i) $F_s[g(x)\cos ax] = \frac{1}{2}[F_s(\lambda + a) + F_s(\lambda - a)]$

(ii) $F_s[g(x)\sin ax] = \frac{1}{2}[F_c(\lambda - a) - F_s(\lambda + a)]$

(iii) $F_c[g(x)\cos ax] = \frac{1}{2}[F_c(\lambda + a) + F_c(\lambda - a)]$

(iv) $F_c[g(x)\sin ax] = \frac{1}{2}[F_s(\lambda + a) - F_s(\lambda - a)]$

Convolution Theorem :

Definition 2.5 : The convolution of the functions $f(x)$ and $g(x)$ is denoted by $f(x) * g(x)$ and defined by

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u) g(x - u) du$$

Theorem 2.5.1: The Fourier transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of the Fourier transforms of $f(x)$ and $g(x)$. In symobls,

$$F[f)x) * g(x) = F[f(x)]F[g(x)] = F(\lambda)G(\lambda)$$

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Proof : We have by definition of convolution,

$$f(x)*g(x)=\int_{-\infty}^{\infty} f(u)g(x-u)du$$

$$\therefore \qquad F[f(x)*g(x)]=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right]e^{-i\lambda x}dx$$

$$=\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right]du \qquad \text{By changing the order of integration}$$

$$=\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(t)e^{-i\lambda(t+u)}dt\right]du, \quad \text{by } x=t+u$$

$$\left[\int_{-\infty}^{\infty} f(u)e^{-i\lambda(t+u)}dt\right]\left[\int_{-\infty}^{\infty} g(t)e^{-i\lambda t}dt\right]=\left[\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx\right]\left[\int_{-\infty}^{\infty} g(x)e^{-i\lambda x}dx\right]$$

$$=F[f(x)]F[g(x)]=F(\lambda)G(\lambda)$$

Remarks : The following important properties of convolution can be proved easily:

1. $f(x)*g(x)=g(x)*f(x)$
2. $f(x)*\{g(x)*h(x)\}=\{f(x)*g(x)\}+h(x)$
3. $f(x)*\{g(x)+h(x)\}=f(x)*g(x)+f(x)+f(x)*h(x)$

i.e., the convolution obeys the commutative, associative and distributive laws of algebra.

3. FINITE FOURIER TRANSFORMS

If a function $f(x)$ is defined in the finite interval $0<x<1,$ and satisfies the Dirichlet's conditions, then it can be expressed as half range cosine or sine series. Using these representations, we define finite Fourier sine and cosine transforms for given function $f(x)$.

Finite Fourier Cosie Transform 3.1. : If a function $f(x)$ is defined in the interval $0\le x\le 1,$ then half range cosine series for $f(x)$ is given by

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_0 \cos\frac{n\pi x}{L} \tag{3.11}$$

where, $$a_0 = \frac{2}{L}\int_0^L f(x)dx \tag{3.12}$$

and $$a_0 = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \tag{3.13}$$

The finite Fourier cosine transform of $f(x)$ is defined as

$$F_c[f(n)] = \int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

where n is an integer

The function $f(x)$ is then called the inverse finite Fourier cosine transform of $F_c[f(n)]$ which is given by

$$f(x) - \frac{1}{K}F_s(0) + \frac{2}{L}\sum_{n-1}^{\infty} F_c[f(u)]\cos\frac{n\pi x}{L} \tag{3.1.5}$$

Note : From (ii) and (iii), we note that

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}F_c[f(n)] \tag{3.1.6}$$

and $$a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{L}F_c(0) \qquad \therefore \frac{a_0}{2} = \frac{1}{L}F_c(0) \tag{3.1.7}$$

On substituting (iv) and (v) in (i), we obtained result (16) for inverse finite Fourier transform.

Finite Fourier Sine Transform 3.2. : If a function $f(x)$ is defined in the interval $0 \le x \le L,$ then half range sine series is given by

$$f(x) = \sum_{n-1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{3.2.1}$$

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where, $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \quad (3.2.2)$$

The **finite Fourier sine transform** is defined as

$$F_s[f(n)] = \int_0^L f(x)\sin\frac{n\pi x}{L}dx \quad (3.2.3)$$

where n is integer

The function $f(x)$ is then called the **inverse finite Fourier sine transform** of $F_s[f(n)]$ which is given by

$$f(x) = \frac{2}{L}\sum_{n=1}^{\infty} F_s[f(n)\sin\frac{n\pi x}{L} \quad (3.2.4)$$

Note : From result (ii), we note that

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}F_s[f(n)] \quad (3.2.5)$$

On substituting b_n from (iii) in (i) we, obtained result (9) for inverse finite Fourier sine transform.

Remark : The finite Fourier cosine and sine transforms can also be denoted by C(n) and S(n) respectively.

Examples 3.4 :

1) Find finite Fourier Cosine transfomrs of $f(x) = 2x \quad for \quad 0 \le x \le 2$

Sol^n : $F_c[f(n)] = \int_0^L f(x)\cos\frac{n\pi x}{L}dx = \int_0^2 2x)\cos\frac{n\pi x}{L}dx$

$$= \left[2x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{4}\right) - 2\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{4}\right)\right]_0^2$$

$$= 2(2)\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right) + 2\left(\frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right) - \frac{8}{n^2\pi^2}\cos 0$$

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$$=\frac{8}{n\pi}\sin\frac{n\pi}{2}-\frac{8}{n^2\pi^2}$$

At $n=0$ then $F[f(n)]$ becomes indeterminate

$$F_c(0)=\int_0^2 2x\,dx=\left[x^2\right]_0^2=4-0=4$$

$\therefore$ Finite cosine transform of $f(x)$ is

$$F_c(f(n))=\frac{8}{n\pi}\sin\frac{n\pi}{2}-\frac{8}{n^2\pi^2} \quad \text{and} \quad F_c(0)=4$$

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