ROLE OF SOME SOLUTIONS OF DIFFERENCE EQUATIONS AND DIFFERENCE INEQUALITIES

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Abstract

In this paper we discuss Some solutions of difference Equations and Difference inequalities in fixed point theory

INTRODUCTION. :

Difference Equation is a relation involving differences, It is also known as recurrence relation. Application of difference equation in many branches such as Social Science, Economics, Dairy science, Agriculture etc.

SOME DEFINITIONS:

a) Difference operator :

Let y(k), be a function real or complex variable k. Difference operator Δ is defined as

$\Delta\, y(k) = y(k+1) - y(k)$

e.g. let $y(x) = x^2$

$y(x+1) = (x+1)^2$

$\Delta y(x) = (x+1)^2 - x^2 = 2x+1$

Fixed Point:-

Let X be any non-empty set. The function

F: $X \to X$ is defined as $f(x) = x, \quad x \in X$

It is known as fixed point in x.

Lipschitzian:- suppose (X, d) be any metric space.

A function $f: X \to X$ is said to be lipschitzian if there exists a constant $\propto \geq 0$ such that

$d(F(x), F(y)) \leq \propto d(x, y)$. for all $x, y \in X$ …..(1)

The smallest $\propto$ for which (1) hold it is said to be lipschitzian constant for f. it is denoted by L.

i) If $L < 1$ then f is called contraction.

ii) $L = 1$ then f is called non expansive.

Theorem 1: let (X, d) be a compact metric space

With $F : X \to X$ satisfying

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d(F(x), F(y)) < d(x, y), for all x, y ∈ X and x ≠ y

then F has a fixed point in X.

Proof: let us suppose that x, y ∈ X. by definition of

Fixed point $F(x) = x$ and $F(y) = y$

$$\therefore d(x,y) = d\,(F(x),F(y) \leq L\,d(x,y)$$

$$\Rightarrow d(x,y) = 0$$

$$\therefore \mathbf{x} = \mathbf{y}$$

Now to prove existence of the map $x \rightarrow (x, F(x))$ attains

its minimum say $x_0 \in X.$ by def^n $F(x_0) = x_0$

$$d\left(F(F(x_0)), F(x_0)\right) < d(F(x_0), x_0)$$

This is not done. Our supposition is wrong

∴F has a unique fixed point in X.

Theorm2:

let C be a non empty, closed and bounded convex set in a (real) Hilbert space H. then each non expansive map $F: C \rightarrow C$ has at least one fixed point.

Proof: - Suppose C is a non empty, closed, convex subset of a normed linear space E

Let $O \in C$ for any $x_0 \in C$ such that $x_0 = 0$

Assume that $F(0) \neq 0$. For each n=2,3…..

Now

$$F_n = \left(1 - \frac{1}{n}\right), \; F : C \rightarrow C$$

Is contraction by theorem 1 then there exists a unique point $x_n \in C$ such that

$$x_n = F_n(x_n) = \left(1 - \frac{1}{n}\right) F(x_n)$$

i.e. $\|x_n - F(x_n)\| = \frac{1}{n}\|F(x_n)\| \leq \frac{1}{n}\delta(c)$.

Where $\delta(c)$ is diameter of C. for $\eta \in \{2,3,\ldots\}$

$$Q_n = \left\{x \in C : \; \|x - F(x)\| \leq \frac{1}{n}\delta(c)\right\}.$$

$$Q_2 = \left\{x \in C : \; \|x - F(x)\| \leq \frac{1}{2}\delta(c)\right\}$$

$$Q_3 = \left\{x \in C : \; \|x - F(x)\| \leq \frac{1}{3}\delta(c)\right\}$$

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$$Q_4 = \left\{x \in C : \|x - F(x)\| \le \frac{1}{4}\delta(c)\right\}$$

i.e. $Q_2 \supseteq Q_3 \supseteq Q_4 \dots\dots \supseteq Q_n \supseteq \cdots$

is a decreasing sequence of a non empty closed set

$d_n = \inf\{\|x\| : x \in Q_n\}$ and

Q_n is a decreasing then

$d_2 \le d \le d_4 \dots\dots \le d_n \le \cdots$ with $d_i \le \delta(c)$

For each $i \in \{2,3,\dots\}$

Consequently $d_n \to d$ with $d \le \delta(c)$

Now $A_n = Q8n^2 \cap \overline{B(0, d + \frac{1}{n})}$

Where $B\left(0, d + \frac{1}{n}\right) = \left\{x \in H : \|x\| < d + \frac{1}{n}\right\}$

A_n is decreasing sequence of closed non empty set

Now we have to prove that $\lim_{n\to\infty} \delta(A_n) = 0$

Let $u, v \in A_n$ then by theorem

$\|u - 0\| < d + \frac{1}{n}$ and $\|v - 0\| < d + \frac{1}{n}$ ……….(1) but $u, v \in Q8n^2$ then

$\|u - F(u)\| \le \frac{1}{8n^2}\delta(c)$ and $\|v - F(v)\| \le \frac{1}{8n^2}\delta(c)$

We knew the fact H is a Hilbert space and $C \le H$ be a bounded and $F : C \to C$ a non expansive map. Suppose $x \in C, y \in C$ there exist a point a such that $a = \frac{x+y}{2} \in C \Rightarrow \|a - F(a)\| \le 2\sqrt{\epsilon}\sqrt{2\delta(c)}$

$$\therefore \left\|\frac{u+v}{2} - F\left(\frac{u+v}{2}\right)\right\| \le 2\sqrt{2\delta(c)}\sqrt{\frac{1}{8n^2}\delta(c)} = \frac{1}{n}\delta(c)$$

$$\therefore \frac{u+v}{2} \in Q_n \text{ and } \left\|\frac{u+v}{2} - 0\right\| \ge d_n \dots\dots\dots\dots\dots\dots (2)$$

We know the fact. "H is a Hilbert space with $u, v \in H$ and r, R be a constant with $o \le r \le R$ it there exists $x \in H$ with $\|u - x\| \le R$; $\|v - x\| \le R$ and $\left\|\frac{u+v}{2} - x\right\| \ge r$

Then

$\|u - v\| \le 2\sqrt{R^2 - r^2}$ …………… (3)

By (1), (2), (3) we conclude that

$$\|u - v\| \le 2\sqrt{(d + \frac{1}{n})^2 - d_n^2} = 2\sqrt{d^2 + d\frac{1}{n} + \frac{1}{n^2} - d_n^2}$$

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$$\delta(A_n) \leq 2\sqrt{\frac{2d}{n} + \frac{1}{n^2} + (d^2 - d_n^2)}$$

Therefore, $\lim_{n\to\infty} \delta(A_n) = 0$

By contor's theorem (Applied$\{A_n\}_{n=2}^{\infty}$)

Guarantees and existence of $x_0 \in \cap_{n=2}^{\infty} A_n$

$$x_0 \in \bigcap_{n=2}^{\infty} Q8n^2$$

$\therefore \|x_0 - F(x_0)\| \leq \frac{\delta(c)}{8n^2}$ (for all $n \in \{2, 3, \ldots\}$)

$\therefore \|x_0 - F(x_0)\|$=0

Hence the proof

Example:-

Let H be a bounded, closed, convex subset of uniformly convex Banach space X and $F: H \to X$ be a non expansive map with

$\text{Inf}\{\|x - F(x)\|: x \in H\} = 0$ show that F has a fixed point in H.

Solution:-

Suppose $F: H \to X$ be a non expansive map then there exists $\{u_n\}, \{v_n\}$ in H such that

$x_n = \frac{u_n+v_n}{2} \in H$; $\|z_n - F(z_n)\| < 2\sqrt{\epsilon}\sqrt{2\delta(H)}$

$\lim_{n\to\infty}\|u_n - F(u_n)\| = 0$ and $\lim_{n\to\infty}\|v_n - F(v_n)\| = 0$

By theorem

$\left\|\frac{u+v}{2} - F\left(\frac{u+v}{2}\right)\right\| = \frac{1}{n}\delta(H)$

$$d_n = \inf\{\|x - F(x)\|: x \in Q_n\}$$

Where $Q_n = \left\{x \in H: \|x - F(x)\| \leq \frac{1}{n}\delta(H)\right\}$

$$\left\|\frac{u + v}{2} - 0\right\| \geq d_n$$

$$\|u - v\| \leq 2\sqrt{(d + \frac{1}{n})^2 - d_n^2}$$

$$\delta(A_n) \leq 2\sqrt{\frac{2d}{n} + \frac{1}{n^2} - (d^2 - d_n^2)}$$

$\lim_{n\to\infty} \delta(A_n) = 0$ **(By contor's theorem)**

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$$x_0 \in \bigcap_{n=2}^{\infty} A_n$$

$$\{\|x - F(x)\| : x \in H\} = 0$$

$\Rightarrow x = F(x)$ it is fixed point in H.

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