ROLE OF SOME SOLUTIONS OF DIFFERENCE EQUATIONS AND DIFFERENCE INEQUALITIES

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Abstract

In this paper we discuss Some solutions of difference Equations and Difference inequalities in fixed point theory

INTRODUCTION.:

Difference Equation is a relation involving differences, It is also known as recurrence relation. Application of difference equation in many branches such as Social Science, Economics, Dairy science, Agriculture etc.

SOME DEFINITIONS:

a) Difference operator :

Let y(k), be a function real or complex variable k. Difference operator $\Delta$ is defined as

$\Delta y(k) = y(k+1) - y(k)$

e.g. let $y(x) = x^2$

$y(x+1) = (x+1)^2$

$\Delta y(x) = (x+1)^2 - x^2 = 2x+1$

Fixed Point:

Let X be any non-empty set. The function $F: X \rightarrow X$ is defined as $f(x) = x, \ x \in X$

It is known as fixed point in x.

Lipschitzian: - suppose (X, d) be any metric space.

A function $f: X \rightarrow X$ is said to be lipschitzian if there exists a constant $\alpha \geq 0$ such that

$d (F(x), F(y)) \leq \alpha d(x, y) \ for \ all \ x, y \in X \ ....(1)$

The smallest $\alpha$ for which (1) hold it is said to be lipschitzian constant for f. it is denoted by L.

i) $L < 1$ then f is called contraction.

ii) $L = 1$ then f is called non expansive.

Theorem 1: let (X, d) be a compact metric space

With $F : X \rightarrow X$ satisfying
\[ d(F(x), F(y)) < d(x, y), \quad \text{for all } x, y \in X \text{ and } x \neq y \]

then F has a fixed point in X.

Proof: let us suppose that x, y \in X. by definition of

Fixed point \( F(x) = x \) and \( F(y) = y \)

\[ \therefore d(x, y) = d(F(x), F(y)) \leq L d(x, y) \]

\[ \Rightarrow d(x, y) = 0 \]

\[ \therefore x = y \]

Now to prove existence of the map \( x \to (x, F(x)) \) attains its minimum say \( x_0 \in X \) by def \( F(x_0) = x_0 \)

\[ d\left(F(F(x_0)), F(x_0)\right) < d(F(x_0), x_0) \]

This is not done. Our supposition is wrong

\[ \therefore F \text{ has a unique fixed point in } X. \]

Theorm 2:

let C be a non empty, closed and bounded convex set in a (real) Hilbert space H. then each non expansive map \( F: C \to C \) has at least one fixed point.

Proof: - Suppose C is a non empty, closed, convex subset of a normed linear space E

Let \( O \in C \) for any \( x_0 \in C \) such that \( x_0 = 0 \)

Assume that \( F(0) \neq 0 \). For each \( n=2,3, \ldots \)

Now

\[ F_n = \left(1 - \frac{1}{n}\right), \quad F: C \to C \]

Is contraction by theorem 1 then there exists a unique point \( x_n \in C \) such that

\[ x_n = F_n(x_n) = \left(1 - \frac{1}{n}\right) F(x_n) \]

i.e. \( \|x_n - F(x_n)\| = \frac{1}{n}\|F(x_n)\| \leq \frac{1}{n} \delta(c). \)

Where \( \delta(c) \) is diameter of C. for \( \eta \in \{2,3,\ldots\} \)

\[ Q_n = \left\{ x \in C : \|x - F(x)\| \leq \frac{1}{n} \delta(c) \right\}. \]

\[ Q_2 = \left\{ x \in C : \|x - F(x)\| \leq \frac{1}{2} \delta(c) \right\} \]

\[ Q_3 = \left\{ x \in C : \|x - F(x)\| \leq \frac{1}{3} \delta(c) \right\} \]
\[ Q_4 = \left\{ x \in C : \| x - F(x) \| \leq \frac{1}{4} \delta(c) \right\} \]

i.e. \( Q_2 \supseteq Q_3 \supseteq Q_4 \ldots \supseteq Q_n \supseteq \ldots \)

is a decreasing sequence of a non empty closed set

\[ d_n = \inf \{ \| x \| : x \in Q_n \} \]

and \( Q_n \) is a decreasing then

\[ d_2 \leq d \leq d_4 \ldots \leq d_n \leq \ldots \] with \( d_i \leq \delta(c) \)

For each \( i \in \{2,3,\ldots\} \)

Consequently \( d_n \to d \) with \( d \leq \delta(c) \)

Now \( A_n = Q8n^2 \cap B(0, d + \frac{1}{n}) \)

Where \( B \left( 0, d + \frac{1}{n} \right) = \left\{ x \in H : \| x \| < d + \frac{1}{n} \right\} \)

\( A_n \) is decreasing sequence of closed non empty set

Now we have to prove that \( \lim_{n \to \infty} \delta(A_n) = 0 \)

Let \( u, v \in A_n \) then by theorem

\[ \| u - 0 \| < d + \frac{1}{n} \quad \text{and} \quad \| v - 0 \| < d + \frac{1}{n} \] \[ \quad \ldots \ldots \text{(1)} \] but \( u, v \in Q8n^2 \) then

\[ \| u - F(u) \| \leq \frac{1}{8n^2} \delta(c) \quad \text{and} \quad \| v - F(v) \| \leq \frac{1}{8n^2} \delta(c) \]

We knew the fact \( H \) is a Hilbert space and \( C \leq H \) be a bounded and \( F : C \to C \) a non expansive map. Suppose \( x \in C, y \in C \) there exist a point \( a \) such that

\[ a = \frac{x+y}{2} \in C \Rightarrow \| a - F(a) \| \leq 2\sqrt{\epsilon} \sqrt{2\delta(c)} \]

\[ \therefore \frac{\| u + v \|}{2} - F \left( \frac{u + v}{2} \right) \leq 2\sqrt{2\delta(c)} \sqrt{\frac{1}{8n^2}} \delta(c) = \frac{1}{n} \delta(c) \]

\[ \therefore \frac{u+v}{2} \in Q_n \quad \text{and} \quad \frac{\| u + v \|}{2} - 0 \geq d_n \quad \ldots \ldots \text{(2)} \]

We know the fact. “H is a Hilbert space with \( u, v \in H \) and \( r, R \) be a constant with \( 0 \leq r \leq R \) it there exists \( x \in H \) with \( \| u - x \| \leq R \); \( \| v - x \| \leq R \) and \( \frac{\| u + v \|}{2} - x \geq r \)

Then

\[ \| u - v \| \leq 2\sqrt{R^2 - r^2} \] \[ \quad \ldots \ldots \text{(3)} \]

By (1), (2), (3) we conclude that

\[ \| u - v \| \leq 2 \sqrt{\left( d + \frac{1}{n} \right)^2 - d_h^2} = 2 \sqrt{d^2 + d \frac{1}{n} + \frac{1}{n^2} - d_h^2} \]
\[\delta(A_n) \leq 2 \sqrt{\frac{2d}{n} + \frac{1}{n^2} + (d^2 - d_n^2)}\]

Therefore, \(\lim_{n \to \infty} \delta(A_n) = 0\)

By contor’s theorem (Applied\(\{A_n\}_{n=2}^\infty\))

Guarantees and existence of \(x_0 \in \bigcap_{n=2}^{\infty} A_n\)

\[x_0 \in \bigcap_{n=2}^{\infty} Q8n^2\]

\[\therefore \|x_0 - F(x_0)\| \leq \frac{\delta(c)}{8n^2} \quad (\text{for all } n \in \{2, 3, \ldots\})\]

\[\therefore \|x_0 - F(x_0)\| = 0\]

**Hence the proof**

**Example:**

Let \(H\) be a bounded, closed, convex subset of uniformly convex Banach space \(X\) and \(F : H \to X\) be a non expansive map with

\[\text{Inf} \{\|x - F(x)\| : x \in H\} = 0\]

show that \(F\) has a fixed point in \(H\).

**Solution:**

Suppose \(F : H \to X\) be a non expansive map then there exists \(\{u_n\}, \{v_n\}\) in \(H\) such that

\[x_n = \frac{u_n + v_n}{2} \in H; \quad \|z_n - F(z_n)\| < 2\sqrt{\varepsilon \sqrt{2\delta(H)}}\]

\[\lim_{n \to \infty} \|u_n - F(u_n)\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|v_n - F(v_n)\| = 0\]

By theorem

\[\left\|\frac{u + v}{2} - F\left(\frac{u + v}{2}\right)\right\| \leq \frac{1}{n} \delta(H)\]

\[d_n = \text{inf} \{\|x - F(x)\| : x \in Q_n\}\]

Where \(Q_n = \{x \in H : \|x - F(x)\| \leq \frac{1}{n} \delta(H)\}\)

\[\left\|\frac{u + v}{2} - 0\right\| \geq d_n\]

\[\|u - v\| \leq 2 \sqrt{(d + \frac{1}{n})^2 - d_n^2}\]

\[\delta(A_n) \leq 2 \sqrt{\frac{2d}{n} + \frac{1}{n^2} - (d^2 - d_n^2)}\]

\[\lim_{n \to \infty} \delta(A_n) = 0\]  \hspace{1cm} (By contor’s theorem)
\[ x_0 \in \bigcap_{n=2}^{\infty} A_n \]
\[ \{ \| x - F(x) \| : x \in H \} = 0 \]

\[ \Rightarrow x = F(x) \text{ it is fixed point in } H. \]

REFERENCE:

12. Difference equation by Kelley and peterson new academic press.