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EXISTENCE AND UNIQUENESS OF SOLUTION OF THIRD ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACE

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Abstract: In this chapter we proved the existence and uniqueness of solution of third order nonlinear ordinary differential equations with initial conditions by using fixed point theory in Banach Space. Some examples are also discussed.

Keywords: Nonlinear, contraction mapping, fixed point, Banach Space

1. INTRODUCTION

CONTRACTION MAPPING ON A BANACH SPACE:

Let X be a Banach space and let $F : X \to X$ then we say that F is a contraction mapping if for all $a, b \in X$ such that

$$||F(a) - F(b)|| \leq \alpha||a - b|| \qquad (1.1)$$

where α is some constant s.t. $0 \leq \alpha < 1$. A "contraction mapping" draws between any two closer points.

EXAMPLE :

Let $X = \alpha([0, 1/2])$ with the suprermum norm and

$$F(h)(t) = \int_0^t h(v)dv$$

then for all $t \in [0, 1/2]$ and $h, k \in X$, we have

$$|F(h)(t) - F(k)(t)| = \left|\int_0^t (h-k)\right| \leq \int_0^t ||h-k||_\infty = t||h-k||_\infty \leq \tfrac{1}{2}||h-k||_\infty.$$

Because this inequality holds for all $t \in [0, 1/2]$ then clearly we have

$$||F(h) - F(k)||\infty \leq \tfrac{1}{2}\ ||h - k||_\infty,$$

Hence, F is a contraction mapping with constant $\alpha = ½$.

THEOREM (THE CONTRACTION MAPPING PRINCIPLE)

Let X be a Banach space on which F is a contraction mapping. Then there is unique fixed point of F in X, i.e., there exists one and only one solution of the equation

$$a = F(a), \text{ for } a \in X$$

2. APPLICATION TO DIFFERENTIAL EQUATIONS

Consider the general equation of first order

$$y'(s) = f(s, y(s)), \qquad (1.2)$$

where $f(s, y)$ is a function of two variables. For example, if $f(s, y) = s + s^2y^2$ then the differential equation is $y'(s) = s + sy^2(s)$ with initial condition $y(s_0) = y_0$. We have to find a function $y(s)$ which satisfies the differential equation with given initial condition. The function is continuous and defined on the interval $[s_0, s_0 + \sigma)$ and differentiable on the open interval $(s_0, s_0 + \sigma)$ for some $\sigma > 0$

The following theorem shows that , under the right conditions, states that a differential equation has a unique solution, even if the solution cannot be written down in closed form.

We will consider only functions $f(s,y)$ which satisfy the following property : It is require that f will be differentiable in y and

$$\left|\frac{\partial f}{\partial y}(s,y)\right| \leq C \qquad (1.3)$$

For all s in some interval $[s, s_0 + \sigma]$ and all y in $[y_0 - \sigma_2, y_0 + \sigma_2]$, where σ_1 and σ_2 are some positive constants. The constant C must be independent of s and y. The simple condition in which this is guaranteed is that $\frac{\partial f}{\partial y}(s,y)$ is continuous for $(s,y) \in [s_0, s_0 + \sigma_1] \times [y_0 - \sigma_2, y0 + \sigma_2]$ since a continuous function is bounded on a compact set. By considering these conditions we can prove, Existence-Uniqueness theorem for first order nonlinear differential equations which is given below.

***Existence-Uniqueness Theorem*:** Consider the function $f(s,y)$ be continuous and satisfy the bound (1.3). Then the differential equation (1.2) with initial condition $y(s_0) = y_0$ has a unique solution which is continuous on some interval $[s_0, s_0 + \sigma]$ and differentiable $(s_0, s_0 + \sigma)$, where $\sigma > 0$.

Now we prove this above theorem for third order nonlinear ordinary differential equations.

3. EXISTENCE-UNIQUENESS THEOREM

Consider the function $f(s,y,y',y'')$ is continuous and satisfying the bound

$$\left|\frac{\partial^3 f}{\partial y^3}(s,y,y',y'')\right| \leq C \qquad (1.3^*).$$

Then the differential equation

$$\left.\begin{aligned} & y'''(s) = f(s, y(s), y'(s), y''(s)) \\ & \text{with initial conditions } y(s_0) = y_0,\ y'(s_0) = y'_0, \qquad (1.2^*) \\ & y''(s_0) = y_0'' \end{aligned}\right\}$$

has a unique solution which is continuous on some interval $[s_0, s_0 + \sigma]$ and differentiable $(s_0, s_0 + \sigma)$, where $\sigma > 0$.

PROOF OF EXISTENCE-UNIQUENESS THEOREM

We first rewrite the given differential equation in the form of an integral equation. Note that if $y''(s)$ is continuous for $s_0 \leq s \leq s_0 + \sigma_1$ then $(s, y(s), y'(s), y''(s))$ is continuous, hence $y''(s) = f(s, y(s), y'(s), y''(s))$ must also be continuous for $s_0 \leq s \leq s_0 + \sigma_1$. To get result we can integrate both sides of differential equation in (1.2*) for $u = s_0$ to $u = s < s_0 + \sigma_1$ and then applying the Fundamental Theorem of Calculus to estimate

$$y''(s) = y_0'' + \int_{s_0}^{s} f(u, y''(u))du. \qquad (1.4)$$

Conversely, any solution of equation (1.4) which is continuous is also a solution of the original differential equation (1.2*), the right hand side of equation (4.4) must be differentiable in s since by the Fundamental Theorem of Calculus and hence we can differentiating both sides of equation (1.4) with respect to s to obtain differential equation. Also, putting $s = s_0$ in the equation (1.4) gives $y''(s_0) = y_0''$. Any differentiable solution $y''(s)$ to (1.2*) with initial condition $y''(s_0) = y_0''$ is necessarily a continuous solution to equation (1.4) and vice-versa. Now we will prove that there exists a unique continuous solution to equation (1.4) on some interval $s_0 \leq s \leq s_0 + \sigma$.

Let us consider proof of the theorem as a simple special case, to get the following idea. Consider that the bound (1.3*) is satisfied with "$\sigma_2 = \infty$", that is , for all y. Suppose any two real numbers r_1 and r_2 then it follows from the Mean Value Theorem that

$$\frac{\partial^2 f(u,r_1) - \partial^2 f(u,r_2)}{r_1 - r_2} = \frac{\partial^3 f}{\partial y^3}(u,v)$$

For some v between r_1 and r_2.

On taking the modulus values of both sides of the above equation and using the bound (1.3*) proves that

$$|\partial^2 f(u,r_1) - \partial^2 f(u,r_2)| \leq C\,|r_1 - r_2| \qquad (1.5)$$

for real numbers r_1, r_2 and all $u \in [s_0, s_0 + \sigma_1]$

We defined an operator $F(y)$ on $C([s_0, s_0 + \sigma_1])$ by

$$F(y)(s) = y_0'' + \int_{s_0}^{s} f(u, y''(u))du$$

By the fundamental theorem of calculus, F really does turn continuous functions into continuous in fact, differentiable functions. The discussion shows the equivalence of equations (1.2) and (1.4) any fixed point for F gives a solution of $y'''(s) = f(s, y(s), y'(s), y''(s))$ with $y''(s_0) = y_0''$. Now we will prove the existence of this fixed point.

By taking use of equation (1.5) we can obtain, for any two functions y_1 and y_2 in

$C([s_0, s_0 + \sigma_1])$, we have

$$|F(y_1)(s) - F(y_2)(s)| = \left|\int_{s_0}^{s} f(u, y_1(u)) - f(u, y_2(u))du\right|$$

$$\leq \int_{s_0}^{s} f(u, y_1(u)) - f(u, y_2(u))du$$

$$\leq C\int_{s_0}^{s} |y_1(u) - y_2(u)|du \quad \text{(by equation (4.5))}$$

$$\leq C||y_1 - y_2||\infty \int_{s_0}^{s} du$$

$$= C||y_1 - y_2||_\infty\ (s - s_0)$$

We choose s close to s_0 such that, we get $C(s - s_0) < 1$ that is, $s < s_0 + \frac{1}{C}$, then F become a contraction mapping. If let us consider $\sigma = min\,(\sigma_1, \frac{1}{C})$ and $I = [s_0, s_0 + \sigma]$. The operator F on C (I) is a contraction and thus equation (1.4) must have a unique fixed point $y''(s)$.

The fundamental theorem of calculus shows that the right hand side of equation (1.4) is differentiable in s, thus $y''(s)$ is also derivable. As substituting $s = s_0$ in the integral equation, it is easily observed that $y''(s_0) = y_0''$ and then differentiating both sides proves that $y'''(s) = f(s, y(s), y'(s), y''(s))$. If the bound (1.3*) holds for all y ,this completes the proof the theorem.

EXAMPLE:

Now we apply the above theorem to the differential equation

$y'''(s) = s^2 + sy^4(s)$ with initial condition $y''(0) = 2$

Here we compute,

$$\frac{\partial f}{\partial y} = 4sy^3\ ;$$

$$\frac{\partial^2 f}{\partial y^2} = 12sy^2\ ;$$

$$\frac{\partial^3 f}{\partial y^3} = 24sy\ ; \text{ which is continuous for all } s \text{ and } y$$

and therefore it is bounded by a constant on any compact set of the form $0 \leq s \leq \sigma_1$, $2 - \sigma_2 < y \leq 2 + \sigma_2$. For example, by taking $\sigma_1 = \sigma_2 = 1$ and therefore we have, $\frac{\partial^3 f}{\partial y^3} = 72$. Thus given differential equation satisfies the conditions of above theorem, therefore this differential equation must have a unique solution for $s \in (0, \sigma)$, for some $\sigma > 0$, with initial condition $y''(0) = 2$.

4. CONCLUSION

In the above result, we proved differential equation has a solution on some interval is said to be a local existence result, this is opposite to a global existence result in which the differential equation has a solution for all time.

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