GLOBAL EXISTENCE AND BOUNDEDNESS OF SOLUTIONS OF CERTAIN NONLINEAR SUMMATION-DIFFERENCE EQUATION OF SECOND ORDER

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I. INTRODUCTION


It would be noted that Naples Valdes [15] dealt with ordinary summation-difference equation of second order :

\[ \Delta^2 x + a(t)f(t, x, \Delta x) \Delta x + g(t, \Delta x) = \sum_{i=0}^{n-1} C(t, \tau) \Delta x(s) \]

In recent paper Graef and Tunc [8] discussed the continuability, boundedness and square summation of solution to the second order functional summation – difference equation of second order with multiple constant delays.

\[ \Delta^2 x + a(t)f(t, x, \Delta x) \Delta x + g(t, x, \Delta x) + \sum_{i=1}^{n} h_i(x(t - t_i)) = \sum_{s=0}^{t} C(t, s) \Delta x(s) \]

The proof of result in [8] involves the definition of Lyapunov –Krasnovskii type functional. Now consider the following non linear and non autonomous summation-difference equation of second order with multiple constant delays.

\[ \Delta(p(x)\Delta x) + a(t)f(t, x, \Delta x) \Delta x + b(t)g(t, \Delta x) = \sum_{i=0}^{n} C(t, x) \Delta x(t - \tau_i) \]

Which can be written in the system from as.

\[ \Delta x = \frac{y}{p(x)} \]

\[ \Delta y = \sum_{i=0}^{n} C(t, x) \frac{y(s)}{p(x(s))} - a(t)f \left( t, x, \frac{y(t)}{p(x)} \right) \frac{y(t)}{p(x)} - b(t)g \left( t, \frac{y}{p(x)} \right) \]

\[ = \sum_{i=0}^{n} C(t, x) \frac{y(s)}{p(x(s))} - a(t)f \left( t, x, \frac{y(t)}{p(x)} \right) \frac{y(t)}{p(x)} - b(t)g \left( t, \frac{y}{p(x)} \right) \]

\[ (A_1) \quad p(x) \leq P, \sum_{i=0}^{n} [\Delta P(u)] < \infty \]

\[ (A_2) \quad 0 < m \leq b(t) \leq a(t) \leq M, \]

\[ (A_3) \quad 0 < c_i \leq c_i(t) \leq C_i, \quad \Delta C_i(t) \leq 0 \]

\[ (A_4) \quad g(t) = \frac{g(t, y)}{y} \leq g_1(y \neq 0), \]

\[ (A_5) \quad h_i(0) = 0, \quad 0 < h_i(x \neq 0), \quad [\Delta h_i(x)] \leq R \]

\[ (A_6) \quad \max \left( \sum_{i=0}^{n} C(t, s) \right) + \sum_{u=t}^{w} C(u, t) \leq R \]

\[ (A_7) \quad R + 2 \lambda \tau^* \leq \frac{m}{p_i} \left( f(t, x, y) + g \right) \quad \text{for all} \quad t, x \quad \text{and} \quad y \]

1.2.1 Main Result

Theorem 1.2.1 : Suppose that conditions \((A_1)-(A_6)\) hold. Then all solution of system (2) are continuabale and bounded.
Proof: We define a Lyapunov-Krasovskii functional by

\[ W(t) = W(t, x(t), y(t)) = e^{-\frac{r(t)}{\mu}} \]

\[ V_0(t, x(t), y(t)) = \alpha_x(t) \mid \Delta x(u) \mid \leq \sum_{u=a(t)}^{+\infty} \mid \Delta p(x(u)) \mid < \infty \]

for \( \alpha_x(t) = \min \{ \alpha_0(t), x(t) \} \)

\[ V_1(t) = V_0(t, x(t), y(t)) = \frac{1}{2} y^2 + p(x) \sum_{i=1}^{N} C_i(t) \sum_{s=0}^{k} h_i(s) \]

Where \( k = \frac{1}{2} \min \{ \alpha_0(t), x(t) \} \)

Let \( (x(t), y(t)) \) be a solution of (2). Calculating time difference of the function \( V_0 \) then,

\[ V'_0(t) = \frac{1}{2} y^2 + p(x) \sum_{i=1}^{N} C_i(t) \sum_{s=0}^{k} h_i(s) \leq \frac{1}{2} y^2 + \frac{1}{2} \sum_{i=1}^{N} C_i \delta x^2 \geq k(x^2 + y^2) \]

From assumption (A1), (A2), (A3) if follows that

\[ V'_0(t) \geq \frac{1}{2} y^2 + p(x) \sum_{i=1}^{N} C_i(t) \sum_{s=0}^{k} h_i(s) \geq \frac{1}{2} y^2 + \frac{1}{2} \sum_{i=1}^{N} C_i \delta x^2 \geq k(x^2 + y^2) \]

Let \( (x(t), y(t)) \) be such a solution of system (2) with initial condition \( (x_0, y_0) \). Since Lyapunov-Krasovskii type functional \( W(t) \) is positive definite and decreasing,

\[ W(t) + \frac{1}{2} \frac{\sum_{i=1}^{N} C_i(t)}{\mu} y^2 \leq \frac{1}{2} y^2 + \frac{1}{2} \sum_{i=1}^{N} C_i \delta x^2 \geq k(x^2 + y^2) \]

\[ \leq \frac{1}{2} y^2 + \frac{1}{2} \sum_{i=1}^{N} C_i(t) h_i(s) \frac{1}{2} \sum_{i=1}^{N} C_i(t) h_i(s) \geq k(x^2 + y^2) \]

It is now clear that the time difference of the functional \( W(t) \) defined by (3) along any section of system (2) leads to

\[ W(t) - e^{-\frac{r(t)}{\mu}} W(t) = V_0(t) \frac{d}{dt} + V'_0(t, x(t), y(t)) + V'_0(t, x(t), y(t)) \]

Therefore using (5) and (6) we consider \( \mu = \frac{2k}{\sum_{i=1}^{N} C_i(t)} \) we obtain

\[ \Delta W(t) \leq 0 \]

Since all the functions appearing in the equation (1) are continuous it is obvious that there exist at least one solution of equation (1) defined on \([t_0, t_0 + \delta)\) for some \( \delta > 0 \). Now we have to show that the solution extended to the entire interval \([t_0, \infty)\). We assume that on contrary that there is first time \( T < \infty \) such that the solution exists on \([t_0, T] \) and

\[ \lim_{t \to T} \left( |x(t)| + |y(t)| \right) = \infty \]

Let \( x(t), y(t) \) be such a solution of system (2) with initial condition \( (x_0, y_0) \). Since Lyapunov-Krasovskii type functional \( W(t) \) is positive definite and decreasing, \( \Delta W(t) \leq 0 \) along
the trajectories of the system (2), we can say that $\Delta W(t)$ is bounded $[t_0, T]$ we have

$$W(T, x(T), y(T)) \leq W(t_0, x_0, y_0) = W_0$$

Hence it follows from (3) and (5)

$$x^2(T) + y^2(T) \leq \frac{W_0}{D}$$

Where $D = k \exp(-\gamma(t)\mu^{-1})$ This inequality indicate that $|x(t)|, |y(t)|$ are bounded as $t \to T$. Therefore, we can conclude that $T < \infty$ is not possible, we must have $T = \infty$

Example : We consider the following nonlinear summation difference equation of second order with two constants delays, $\tau_1 > 0, \tau_2 > 0$,

$$\left( 2 + \frac{\sin x}{1 + x^2} \right) \Delta x + \left( 1 + \frac{2}{1 + r} \right) \left( e^{-r} + \text{sin } x + \cos \Delta x + 6 \right) \Delta x$$

$$+ \left( 1 + \frac{1}{1 + r} \right) \left( 3 \Delta x + \frac{\Delta x}{1 + \Delta x} \right) + 2(e + e^{-r}) x(t - \tau_1)$$

$$+ 2(e + e^{-r}) x(t - \tau_2) = \sum_{i=0}^{s} s \Delta x(s) \cdots (7)$$

When we compare equation (7) with equation (1), the existence can be seen of following estimates.

$$p(x) = 2 + \frac{\sin x}{1 + x^2}, \quad 1 \leq p(x) \leq 3$$

$$\sum_{i=0}^{s} |\Delta p(u)| \leq \sum_{i=0}^{s} \left[ \frac{\cos u}{1 + u^2} \right] \leq M$$

$$a(t) = 1 + \frac{2}{1 + r^2}, \quad b(t) = 1 + \frac{1}{1 + r^2}$$

$$m = 1 \leq h(t) \leq a(t) \leq 3 - M$$

$$e_i(t) = e^{t} + e^{-t}, c_i(t) = e^t + e^{-t}$$

$$c_i(t) = e^{t} + e^{-t}, c_i(t) = e^{t} + e^{-t}$$

$$c_1 = e \leq c_i(t) \leq e + 1 \leq c_i$$

$$c_2 = e^t \leq c_i(t) \leq e^t + 1 = c_2$$

$$c_2(t) \leq 0, c_2(t) \leq 0, t \geq 0$$

$$f(t, x, \Delta x) = e^{-t} + \text{sin } x + \cos \Delta x + 6, \quad 4 \leq f(t, x, \Delta x) \leq 9$$

$$f(t, x) = 3 \Delta x + \frac{\Delta x}{1 + x^2}$$

$$g(t, 0) = 0, \quad g_0 = 3 \leq g(t, x) \leq 4 = g_1(y \neq 0)$$

$$h_i(x) = h_i(x) = 2x \frac{h_i(x)}{x} = \frac{h_i(x)}{x} = 2(x \neq 0)$$

$$h_1(0) = h_2(0) = 0$$

$$\sum_{i=0}^{s} c(t, s) + \sum_{i=0}^{s} c(t, s)$$

$$\sum_{i=0}^{s} \left[ \frac{s}{(1 + 2r^2)^2} \beta \frac{t}{(1 + 2r^2)} \right] \leq \frac{3}{8} = R$$

Therefore, all the assumptions of the theorem hold. So we can conclude that all solution of (7) are continuous and bounded.

REFERENCE


