International Journal For Technological Research In Engineering
Volume 8, Issue 2, October-2020 *ISSN (Online): 2347 - 4718*

GLOBAL EXISTENCE AND BOUNDEDNESS OF SOLUTIONS OF CERTAIN NONLINEAR SUMMATION-DIFFERENCE EQUATION OF SECOND ORDER

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I. INTRODUCTION

In recent years, there have been sorrel paper written on the global existence and boundedness of solutions for certain nonlinear difference and summation difference equations of second order with and without delays. Ahmadand Rama Mohan Rao [1], Boxley [2], Byrton [3], Constantin [4] driver [5], Fujimotoa and yamaoka [6], Grace and Lalli [7], Graef and tunc [8], Kalmanovskii and Myshkis [9] Krasovskii [10], Miller [11,12] Mustafa and Rogovchenko [13,14], Nepoles Valdes [15], Ogundare et al, [16], Reissing et al, [17]] Tidke[18], Tiryaki and Zafer [19]] Tunc [20-25]] Tunc and Tunc [26]] Yoshizawa [27]] wu et al. [28] Yin [29]

It would be noted that Napoles Valdes [15] dealt with ordinary summation-difference equation of second order :

$$\Delta^2 x + a(t) f(t, x, \Delta x)\Delta x + g(t, \Delta x) = \sum_{s=0}^{t-1} C(t, \tau)\Delta x(s)$$

In recent paper Graef and Tunc [8] discussed the countinuabielity, boundedness and squore summation of solution to the second order functional summation – difference equation of second order with multiple constant delays.

$$\Delta^2 x + a(t) f(t, x, \Delta x)\Delta x + g(t, x, \Delta x) + \sum_{i=1}^{n} hi\big(x(t - ti)\big) = \sum_{s=0}^{t-1} C(t, s)\Delta x(s)$$

The proof of result in [8] involves the definition of Lyapunov –Kransovkii type functional. Now we consider the following non linear and non autonomous summation-difference equation of second order with multiple constant delays.

$$\Delta(p(x)\Delta x) + a(t) f(t, x, \Delta x)\Delta x + b(t) g(t, \Delta x) + \sum_{i=1}^{n} Ci(t) hi(x(t - \tau i)) = \sum_{s=0}^{t-1} C(t, s)\Delta x(s) \quad (1)$$

Which can be written in the system from as.

$$\Delta x = \frac{y}{p(x)}$$

$$\Delta y = \sum_{s=0}^{t-1} C(t, x)\frac{y(s)}{p(x(s))} - a(t) f\left(t, x, \frac{y}{p(x)}\right)\frac{y}{p(x)} - b(t) g\left(t, \frac{y}{p(x)}\right) \quad (2)$$

$$- \sum_{i=1}^{n} C_i(t) h_i\big(x(t_i)\big) + \sum_{i=1}^{n} C_i(t) \sum_{s=t-x_i}^{t-1} \Delta h_1\left(x(s)\frac{y(s)}{p(x(s))}\right)$$

Where τ_i in $(i$=1,2…….$n)$ are positive constant a,b,c $: R^+ \to R^+$,
$R^+ = (0, \infty)$, $f : R^+ \times R^2 \to R^+$ and $g : R^+ \times R \to R^+$ are continuos functions, $h_i \in C^1 (R_1 R), P \in C^1\big(R, (0, \infty)\big)$, and $C(t,s)$ is continuous function for $0 \le t \le s \le \infty$

The aim of this paper is to give some sufficient conditions to guarantee the global existence and boundedness of solutions of equation (1).This show that novelty and originality of present paper. This paper may also be useful for researchers working on the qualitative behavior of solutions of functional summation-difference equations.

We assume that there are positive constants $\delta_i, \beta_i, \gamma_i, \lambda_i, p, m,$
M, g_0, g_1, C, C_i, R and such τ^* that the following condition hold :

$$(A_1) \le p(x) \le P_1, \sum_{u=-\infty}^{+\infty} [\Delta P(u)] < \infty$$

$$(A_2) 0 < m \le b(t) \le a(t) \le M,$$

$$0 < c_1 \le c_i(t) \le C_i, \quad \Delta C_1(t) \le 0$$

(A_3) $g(t$ $=$ and

$$0 \le g_0 \le \frac{g(t, y)}{y} \le g_1 (y \ne 0),$$

$$(A_4) \quad h_i(0) = 0, \quad 0 < \delta_i \le \beta_i (x \ne 0), \quad [\Delta h_i(x)] \le r_i$$

$$(A_5) \quad \max\left(\sum_{s=0}^{t-1} | C(t, s) | + \sum_{u=t}^{\infty} C(u, t) \right) \le R$$

(A_6) $R + 2\lambda\tau^* \le \frac{m}{p_i} (f (t, x, y) - g$ for all t, x and y

1.2.1 Main Result

Theorem 1.2.1 : Suppose that conditions $(A_1) - (A_6)$ hold. Then all solution of system (2) are continuable and bounded.

Proof : We define a Lyapunov-Krasovskii functional by

$$W(t)=W(t,x(t),y(t))=e\frac{-r(t)}{\mu}$$

$$V_0(t,x,(t),y(t)) \qquad (3)$$

Where μ is a positive constant

$$y(t)=\sum_{s=0}^{t-1}|\theta(s)|=\sum_{s=0}^{t-1}|\Delta x(s)\Delta p(x(s))|$$

$$=\sum_{u=\alpha_1(t)}^{\alpha_2(t)}|\Delta p(x(u))|\le\sum_{u=-\infty}^{+\infty}|\Delta p(x(u))|<\infty$$

for $\theta(t)=\Delta x(t)\Delta p(x(t))=\alpha_1(t)=\min\{x(0),x(t)\}$

$\alpha_2(t)=\max\{x(0),x(t)\}$ and

$$V_0(t)=V_0(t,x(t),y(t))=\frac{1}{2}y^2+p(x)\sum_{i=1}^{n}C_i(t)\sum_{i=0}^{x-1}h_i(s)$$

$$+\sum_{i=1}^{n}\lambda_i\sum_{u=-t_i}^{0}\sum_{s=t+x}^{t-1}y^2(u)+\sum_{s=0}^{t-1}\sum_{u=t}^{\infty}|C(u,s)|y^2(s) \qquad (4)$$

From assumption (A_1), (A_2), (A_4) if follows that

$$V_0(t)\ge\frac{1}{2}y^2+p(x)\sum_{i=1}^{n}C_i(t)\sum_{s=0}^{x-1}h_i(s)$$

$$V_0(t)\ge\frac{1}{2}y^2+p(x)\sum_{i=1}^{n}C_i(t)\sum_{s=0}^{x-1}h_i(s)\ge\frac{1}{2}y^2+\frac{1}{2}\sum_{i=1}^{n}C_i\delta_i x^2\ge k(x^2+y^2) \qquad (5)$$

Where $k=\frac{1}{2}\min\left\{1,\sum_{i=1}^{n}c_i\delta_i\right\}$

Let $(x(t,y(t))$ be a solution of (2). Calculating time difference of the function V_0 then,

$$V_0(t)=y\sum_{s=0}^{t-1}C(t,s)\frac{y(s)}{p(x(s))}-a(t)f(t,x,\frac{y}{p(x)})\frac{y^2}{p(x)}-b(t)g\left(t,\frac{y}{p(x)}\right)y$$

$$+y\sum_{i=1}^{n}C_i(t)\sum_{s=t-\tau_i}^{t-1}\Delta hi(x(s))\frac{y(s)}{p(x(s))}+\theta(t)\sum_{i=1}^{n}c_i(t)\sum_{s=t-x_i}^{x-1}h_i(s)$$

$$V_0(t)=y\sum_{s=0}^{t-1}C(t,s)\frac{y(s)}{p(x(s))}-a(t)f(t,x,\frac{y}{p(x)})\frac{y^2}{p(x)}-b(t)g\left(t,\frac{y}{p(x)}\right)y$$

$$+y\sum_{i=1}^{n}C_i(t)\sum_{s=t-\tau_i}^{t-1}\Delta hi(x(s))\frac{y(s)}{p(x(s))}+\theta(t)\sum_{i=1}^{n}c_i(t)\sum_{s=t-x_i}^{x-1}h_i(s)$$

$$+p(x)\sum_{i=1}^{n}\Delta C_i(t)\sum_{s=0}^{t-1}hi(s)+\sum_{i=1}^{n}(\lambda_i\tau_i)y^2-\sum_{i=1}^{n}\lambda_i\sum_{s=t-x_i}^{t-1}y^2(s)$$

$$+y^2\sum_{u=t}^{\infty}|C(u,t)|-\sum_{s=0}^{t-1}|C(t,s)|y^2(s)$$

By the assumptions (A1), (A6) and the inequality $2|ab|\le(a^2+b^2)$ the following estimates canbe check out.

$$y\sum_{s=0}^{t-1}|C(t,s)\frac{y(s)}{p(x(s))}\le\sum_{s=0}^{t-1}|C(t,s)|y(t)||y(s)|$$

$$\le y^2\sum_{s=0}^{t-1}|C(t,s)|+\sum_{s=0}^{t-1}|C(t,s)|y^2(s)$$

$$-a(t)f\left(t,x,\frac{y}{p(x)}\right)\frac{y^2}{p(x)}-b(t)g\left(t,\frac{y}{p(s)}\right)y\le\frac{-m}{p_1}\left(f\left(f(t,x,\frac{y}{p(x)}\right)+g_0\right)y^2$$

$$y\sum_{i=1}^{n}C_i(t)\sum_{s=t-\tau_i}^{t-1}\Delta h_i(x(s))\frac{y(s)}{p(x(s))}\le\sum_{i=1}^{n}C_ir_i\sum_{s=t-\tau_i}^{t-1}|y(t)y(s)|$$

$$\le\Sigma(C_ir_i\tau_i)y^2+\sum_{i=1}^{n}C_ir_i\sum_{s=t-\tau_i}^{t-1}y^2(s)$$

$$\theta(t)\sum_{i=1}^{n}C_i(t)\sum_{s=0}^{x-1}h_i(s)\le\frac{|\theta(t)|}{2}\sum_{i=1}^{n}(C_i\beta_i)x^2$$

$$p(x)\sum_{i=1}^{n}\Delta C_i(t)\sum_{s=0}^{x-1}h_i(s)\le 0$$

From these estimate we obtain, quite readily,

$$\Delta V_0(t)\le\left(R-\frac{m}{p_i}\left(f\left(t,x,\frac{1}{p(x)}\right)+g_0\right)\right)y^2$$

$$+\sum_{i=1}^{n}(C_ir_i\tau_i)y^2+\sum_{i=1}^{n}(C_ir_i\tau_i)y^2+\sum_{i=1}^{n}C_ir_i\sum_{s=t-\tau_i}^{t-1}y^2(s)$$

$$+\sum_{i=1}^{n}(\lambda_i\tau_i)y^2-\sum_{i=1}^{n}\lambda_i\sum_{s=t-\tau_i}^{t-1}y^2(s)+\frac{|\theta(t)|}{2}\sum_{i=t}^{n}(C_i\beta_i)x^2$$

Let $\tau^*=\max\{\tau_1,\tau_2.........\tau_n\}$ and $\lambda=\sum_{i=1}^{n}\lambda_i=\sum_{i=1}^{n}C_ir_i$

Hence, in view of discussion and (A_6) we conclude that

$$\Delta V_0(t)\le(R+2\lambda\tau^*-\frac{m}{p_i}f(t,x,\frac{y}{p(x)}+g_0))y^2+\frac{|\theta(t)|^2}{2}\sum_{i=1}^{n}(C_i\beta_i)x^2$$

$$\le\frac{|\theta(t)|}{2}\sum_{i=1}^{n}(Ci\beta i)x^2 \qquad (6)$$

It is now clear that the time difference of the functional W(t) defined by (3) along any section of system (2) leads that

$$\Delta W(t)=e\frac{-\gamma(t)}{\mu}\left(\frac{|\theta(t)|}{\mu}V_0(t,x(t),y(t))+\Delta V_0(t,x(t),y(t))\right)$$

Therefore using (5) and (6) we consider $\mu=\frac{2k}{\sum_{i=1}^{n}(C_i\beta_i)}$ we obtain

$$\Delta W(t)=e\frac{-\gamma(t)}{\mu}\left(\frac{-|\theta(t)|}{2}\sum_{i=1}^{n}(C_i\beta_i)x^2\frac{|\theta(t)|}{2}\sum_{i=1}^{n}(C_i\beta_i)x^2\right)=0$$

$$\therefore\Delta W(t)\le 0$$

Since all the functions appearing in the equation (1) are continuous it is obvious that there exist at least one solution of equation (1) defined on $[t_0, t_0+\delta]$ for some $\delta>0$. Now we have to show that the solution extended to the entire interval $[t_0,\infty]$ We assume that on contrary that there is first time $T<\infty$ such that the solution exists on $[t_0,T]$ and

$$\lim_{t\to T}(|x(t)|+|y(t)|)=\infty$$

Let $x(t)$, $y(t)$ be such a solution of system (2) with initial condition (x_0,y_0). Since Lyapunov-Krasovskii type functional $W(t)$ is positive definite and decreasing, $\Delta W(t)\le 0$ along the trajectories of the system (2), we can say that $\Delta W(t)$ is bounded $[t_0 T]$ we have

$$W(T, x(T), y(T)) \leq W(t_0, x_0, y_0) = W_0$$

Hence it follows from (3) and (5)

$$x^2(T) + y^2(T) \leq \frac{W_0}{D}$$

Where $D = k\exp(-\gamma(t)\mu^{-1})$ This inequality indicate that $|x(t)|$, $|y(t)|$ are bounded as $t \longrightarrow T$. Therefore, we can conclude that $t < \infty$ is not possible, we must have $T = \infty$

Example : We consider the following nonlinear summation difference equation of second order with two constants delays, $\tau_1 > 0, \tau_2 > 0,$

$$\left(\left(2+\frac{\sin x}{1+x^2}\right)\Delta x\right)+\left(1+\frac{2}{1+t^2}\right)(e^{-t}+\sin x+\cos\Delta x+6)\Delta x$$

$$+\left(1+\frac{1}{1+t^2}\right)\left(3\Delta x+\frac{\Delta x}{1+(\Delta x)^2}\right)+2(e+e^{-t})x(t-\tau_1)$$

$$+2\left(e^2+e^{-t}\right)x(t-T_2)=\sum_{s=0}^{t-1}\frac{s}{(1+2t)^2}\Delta x(s) \quad \ldots \quad (7)$$

When we compare equation (7) with equation (1), the existence can be seen of following estimates.

$$P(x) = 2 + \frac{\sin x}{1+x^2}, \quad 1 \leq p(x) \leq 3$$

$$\sum_{u=-\infty}^{\infty} |\Delta p(u) \leq \sum_{u=-\infty}^{\infty}\left(\left|\frac{\cos u}{1+u^2}\right| + \frac{2u\sin u}{(1+u^2)^2}\right) \leq \pi$$

$$a(t) = 1 + \frac{2}{1+t^2}, \quad b(t) = 1 + \frac{1}{1+t^2}$$

$$m = 1 \leq b(t) \leq a(t) \leq 3 - M,$$

$$c_1(t) = e + e^{-t}, c_2(t) = e^2 + e^{-t}$$

$$c_1(t) = e + e^{-t}, c_2(t) = e^2 + e^{-t}$$

$$c_1 = e \leq c_1(t) \leq e + 1 = c_1$$

$$c_2 = e^2 \leq c_2(t) \leq e^2 + 1 = c_2$$

$$c_1^1(t) \leq 0, c_2^1(t) \leq 0, t \geq 0$$

$$f(t, x, \Delta x) = e^{-\tau} + \sin x + \cos\Delta x + 6, \quad 4 \leq f(t, x, \Delta x) \leq 9$$

$$f(t, \Delta x) = 3\Delta x + \frac{\Delta x}{1+x^2}$$

$$g(t,0) = 0, \quad g_0 = 3 \leq \frac{g(t,x)}{\Delta x} \leq 4 = g_1(y \neq 0)$$

$$h_1(x) = h_2(x) = 2x \quad \frac{h_1(x)}{x} = \frac{h_2(x)}{x} = 2(x \neq 0)$$

$$h_1(0) = h_2(0) = 0$$

$$|\Delta h_1(x)| = |\Delta h_2(x)| = 2(x+1) - 2x = 2$$

$$\therefore \sum_{s=0}^{t-1} |c(t,s)| + \sum_{u=t}^{\infty} |c(u,t)|$$

$$\therefore \sum_{s=0}^{t-1}\left|\frac{s}{(1+2t)^2}\right| ds + \sum_{u=t}^{\infty}\left|\frac{t}{(1+2u)^2}\right| \leq \frac{3}{8} = R$$

Therefore, all the assumptions of the theorem hold. So we can conclude that all solution of (7) are continuouble and bounded.

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