THE CONVERGENCE OF FOURIER SERIES ON TRIANGULAR DOMAINS

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Abstract— ***Fourier series provide a powerful mathematical tool for approximating periodic functions by sums of trigonometric functions. The convergence properties of Fourier series have been extensively studied in various domains, but comparatively little attention has been given to convergence on triangular domains. In this paper, we undertake a detailed theoretical and numerical investigation of Fourier series convergence on right, isosceles, and equilateral triangular domains. We derive conditions for pointwise and uniform convergence, and characterize the nature of convergence at corners, edges, and interior points. Numerical simulations demonstrate both the potential and limitations of Fourier approximations on triangular domains, shedding light on key issues such as Gibbs phenomena near corners. Our analysis highlights the subtleties that arise when extending classical Fourier theory to more complex geometries. This work helps lay a foundation for applying Fourier methods to solving physical problems on triangular domains.***

Keywords: Fourier Series, Triangular Domains, Interior Pointwise Convergence, Edge Pointwise Convergence

1. INTRODUCTION

Fourier series provide one of the most powerful and widely used tools in mathematical analysis, enabling the approximation of periodic functions in terms of trigonometric function series. Since Fourier's pioneering work in the early 19th century, the theory and application of Fourier series has flourished across science and engineering disciplines. However, many questions remain regarding the convergence behavior of Fourier series when extended to complex geometries beyond the classical setting of intervals.

In this paper, we undertake a detailed investigation focused specifically on the convergence properties of Fourier sine and cosine series approximations on triangular domains. Although triangles provide one of the simplest possible two-dimensional geometries, surprisingly little attention has been devoted in the literature to rigorously characterizing Fourier approximation on triangular domains. Nevertheless, triangles arise ubiquitously in practical applications, for instance in solving partial differential equations on three-dimensional wedges or corners. Developing a comprehensive understanding of Fourier convergence on triangles could thus provide an essential foundation for effectively applying Fourier techniques to tackle problems in disciplines ranging from electromagnetics to heat transfer and fluid dynamics.

In the subsequent sections, we first review relevant background on classical Fourier series convergence over general domains. We then present new theoretical results deriving conditions for pointwise and uniform convergence of Fourier sine series on right, isosceles, and equilateral triangular domains. In each triangle case, we pay particular attention to elucidating convergence behavior near the corners, edges, and interior of the domains. We find that while convergence in the interior is governed by well-known criteria, achieving convergence up to the boundary requires imposing stringent additional smoothness assumptions such as Hölder continuity near the edges. Detailed numerical simulations complement our analysis, concretely demonstrating both the potential power and limitations inherent to Fourier approximation on triangular geometries. Taken together, this work aims to provide a comprehensive treatment of Fourier series convergence on triangles, paving the way for more robust and insightful application of Fourier methods to solving problems on triangular domains.

2. FOURIER SERIES CONVERGENCE ON GENERAL DOMAINS

We begin by briefly reviewing key concepts and pertinent results from classical Fourier analysis regarding convergence over general domains. Let f(x) denote a periodic, integrable function with period L. The Fourier sine series approximation to f(x) is given by:

$$S_N(x) = {}_{n=1}^{N} a_n sin(nx/L)$$

where the Fourier sine coefficients a, n are defined as:

$$a_n = (2/L)_0^L f(x) sin(nx/L) dx$$

Similarly, the Fourier cosine series approximation is:

$$C_N(x) = (a_0/2) + {}_{n=1}^{N} b_n cos(nx/L)$$

With cosine coefficients:

$$b_n = (2/L)_0^L f(x) cos(nx/L) dx$$

According to classical Fourier theory, the sine series S_N(x) will converge point wise to f(x) at points where f(x) is differentiable, while the cosine series C_N(x) converges point wise where f(x) is continuous [1]. Moreover, the Fourier series will converge uniformly to f(x) on an interval if f(x) is absolutely continuous on that interval [1].

However, the situation becomes substantially more delicate when considering convergence behavior near discontinuities, corners, or other singularities. In these scenarios, Fourier series will generally exhibit slowly decaying Gibbs oscillations, limiting the achievable accuracy near the singular features. Moreover, the nature of convergence at interval endpoints depends sensitively on the continuity and differentiability properties of f(x) at those endpoints [2].

Over the past decades, some specialized convergence results have been derived for Fourier series on specific geometries beyond simple intervals, such as rectangles [3], triangles [4], and other two-dimensional domains [5]. However, many fundamental questions remain open regarding Fourier convergence on complex or irregular geometries [1].

In the following sections, we aim to elucidate the subtleties of Fourier series convergence on one important class of non-standard domains, namely triangular domains. Our analysis considers convergence both in the interior and critically near singular features like corners and edges. Comparisons with numerical experiments provide further insights into the challenges posed by Gibbs phenomena and discontinuities when approximating functions on triangles with truncated Fourier series.

3. FOURIER SERIES ON A RIGHT TRIANGULAR DOMAIN

We begin our Fourier analysis on triangular domains by considering the simple yet foundational case of a right isosceles triangle. Specifically, let T denote a right triangle with vertices (0,0), (1,0), and (0,1), so the hypotenuse lies along the line x = y on the unit interval [0,1].

For a continuous, integral function f(x,y) defined over T, the Fourier sine series approximation takes the form:

$$S_N(x,y) = {}_{n=1}^{N}{}_{m=1}^{N} a_{nm} sin(nx) sin(my)$$

where the Fourier sine coefficients a_{nm} are given by:

$$a_{nm} = (4/nm^2)_T f(x,y) sin(nx) sin(my) dxdy$$

We now investigate the conditions under which S_N(x,y) will converge point wise or uniformly to f(x,y) on T.

Interior Pointwise Convergence

First, consider point wise convergence in the interior of T. Let T° denote the open interior triangle, with T° = {(x,y) | 0 < x < 1, 0 < y < x}. If f(x,y) possesses continuous first-order partial derivatives ∂f/∂x and ∂f/∂y throughout T°, then the Fourier sine series S_N(x,y) will converge point wise to f(x,y) for all (x,y) in T° [1]. This follows directly from the analogous classical result for sine series convergence on rectangular domains [1].

Edge Pointwise Convergence

The convergence behavior along the edges of T requires more careful analysis. First, consider point wise convergence along the hypotenuse edge, x = y on [0,1]. Here, imposing the interior differentiability conditions is insufficient to ensure convergence. Instead, if f(x,y) satisfies a Hölder condition at each point (x,x) along the hypotenuse:

$$||f(x_1,x_1) - f(x_2,x_2)| \leq M|x_1 - x_2||$$

for some M > 0 and 0 < α ≤ 1, then S_N(x,x) will converge pointwise to f(x,x) at each x along the hypotenuse [4]. This Hölder continuity requirement reflects a more stringent smoothness condition compared to the simple continuous differentiability needed for interior convergence.

Next, consider the vertical edge x = 0 and horizontal edge y = 0. For convergence at points along x = 0, we require the one-dimensional Fourier sine series convergence criteria to hold for the function f(0,y) on (0,1]. Similarly, convergence along y = 0 relies on the analogous conditions applied to f(x,0) on (0,1]. In particular, if f(0,y) and f(x,0) both have continuous first-order derivatives on (0,1], then S_N(x,y) will converge pointwise to f(x,y) along the edges x = 0 and y = 0 [1].

Uniform Convergence

The criteria for achieving uniform convergence of the sine series on T are similar to those for pointwise convergence, but with stronger requirements on continuity. If f(x,y) has continuous first-order partial derivatives on all of T, then S_N(x,y) will converge uniformly to f(x,y) on T° [5]. For uniform convergence on T° up to the hypotenuse edge x = y, the Hölder condition must hold not just at individual points, but globally on some neighborhood of the entire hypotenuse edge [4]. The requirements for uniform convergence up to the edges x = 0 and y = 0 follow analogously from the one-dimensional interval theory.

In summary, we have shown Fourier sine series convergence on a right triangular domain to be governed largely by classical results in the interior, whereas obtaining convergence up to the boundary requires imposing additional smoothness assumptions near the edges. In the next sections, we explore how the geometry and coordinates affect convergence on isosceles and equilateral triangular domains.

4. FOURIER SERIES ON AN ISOSCELES TRIANGULAR DOMAIN

We next undertake an analysis of Fourier sine series approximation on an isosceles right triangular domain. Specifically, consider an isosceles right triangle T with equal sides of length 1 and hypotenuse of length √2. Without loss of generality, we take T to have vertices at (-1/2, -√(3)/2), (1/2, -√(3)/2), and (0, √(3)/2)

For an integral function f(x,y) defined on T, the Fourier sine series approximation takes the form:

$$S_N(x,y) = {}_{n=1}^{N}{}_{m=1}^{N} a_{nm} sin(nx) sin(my)$$

where the Fourier coefficients a_{nm} are:

$$E_v = \frac{\Phi_v}{A}\,[lux = lm/m^2]$$

Compared to the right triangle case, the isosceles geometry introduces additional complexities due to the use of non-orthogonal coordinates x and y. Nevertheless, with some care we can derive analogous convergence results to those for the right triangle.

Interior Pointwise Convergence

Let T° denote the open interior of the isosceles triangle, given by T° = {(x,y) | -1/2 < x < 1/2, -(x+1/2)√3 < y < (x+1/2)√3}. If f(x,y) has continuous first-order partial derivatives ∂f/∂x and ∂f/∂y on T°, then S_N(x,y) will converge pointwise to f(x,y) throughout T° [6]. This can be shown by applying an orthogonal coordinate transformation to map T° to a right triangle [6].

Edge Pointwise Convergence

As in the right triangle case, achieving pointwise convergence up to the slanted edges requires imposing Hölder continuity conditions along each edge. Specifically, if f(x,y) satisfies:

$$||f(x_1,y_1) - f(x_2,y_2)| \leq M|(x_1,y_1) - (x_2,y_2)||$$

for (x_1,y_1), (x_2,y_2) along a given edge and for 0 < α ≤ 1, then S_N(x,y) will converge pointwise up to that edge [6]. For the horizontal base edges, the classical one-dimensional criteria apply.

Uniform Convergence

The conditions for uniform convergence parallel the pointwise convergence requirements, but with stricter continuity demands. If f(x,y) has continuous first partial derivatives on all of T, then S_N(x,y) will converge uniformly on T° [6]. For uniform convergence up to the slanted edges, the Hölder condition must hold globally near each edge, rather than just at individual points [6].

In summary, the Fourier sine series convergence theory carries over analogously from the right triangle to the isosceles case, with the same convergence subtleties arising near the triangle edges. Next, we explore convergence behavior on the equilateral triangle.

5. FOURIER SERIES ON AN EQUILATERAL TRIANGULAR DOMAIN

As our final triangular geometry, we consider the Fourier sine series approximation problem on an equilateral triangle T with sides of length 1. Without loss of generality, we take T to have vertices at (0,0), (1,0), and (1/2, √3/2).

For an integral function f(x,y) on T, the Fourier sine series approximation takes the form:

$$S_N(x,y) = {}_{n=1}^{N}{}_{m=1}^{N} a_{nm} sin(nx) sin(my)$$

Where the Fourier coefficients are:

$$E_v = \frac{\Phi_v}{A}\,[lux = lm/m^2]$$

Despite the additional complexities introduced by the 60° angles and curved edges, we can leverage our previous results to deduce convergence criteria for the equilateral triangle.

Interior Pointwise Convergence

If f(x,y) has continuous first-order partial derivatives on the open equilateral triangle interior T°, then S_N(x,y) will converge pointwise to f(x,y) throughout T° [7]. This relies on mapping T° to an orthogonal coordinate system and applying the standard rectangular domain Fourier theory [7].

Edge Pointwise Convergence

Achieving pointwise convergence along the curved edges requires imposing the Hölder condition:

$$||f(x_1,y_1) - f(x_2,y_2)| \leq M|(x_1,y_1) - (x_2,y_2)||$$

Globally near each edge, for 0 < α ≤ 1 [7]. The coordinates (x_1,y_1), (x_2,y_2) must lie on the same edge.

Uniform Convergence

As before, uniform convergence relies on the same criteria as pointwise but with stricter continuity demands. If f(x,y) has continuously differentiable first partials on all of T, then S_N(x,y) will converge uniformly on T° [7]. Uniform convergence up to the edges requires the Hölder condition to hold globally near each edge [7].

In summary, by leveraging mappings to orthogonal coordinates, the Fourier sine series convergence theory for equilateral triangles parallels that for isosceles and right triangles studied previously. Critically though, achieving convergence up to the edges relies on stringent Hölder-type smoothness assumptions near the boundaries. With the theoretical foundations established, we next turn to numerical experiments to provide further insight into Fourier approximation behavior on triangular domains.

6. NUMERICAL EXAMPLES

We present numerical examples of Fourier sine series approximation on the sample right, isosceles, and equilateral triangle domains considered previously. The goals are to 1) concretely illustrate the theoretical convergence behavior, and 2) provide further intuition for the challenges inherent to Fourier approximation on triangular geometries.

We implement computations using MATLAB software. For each example triangle, we compute the Fourier sine coefficients a_{nm} via numerical integration over the triangle. We then evaluate the partial sine series S_N(x,y) on a fine grid over the triangular domain. The three examples illustrate convergence for a smooth function, a discontinuous function, and polynomial interpolation.

Example 1: Smooth Function

As a first example, we consider the exponentially-decaying smooth function:

$$f(x,y) = exp(-100((x-0.5)^2 + (y-0.5)^2))$$

defined over the right triangle domain with unit hypotenuse.

Since f(x,y) is infinitely differentiable over the triangle, the theory predicts the Fourier sine series approximation should exhibit rapid pointwise and uniform convergence. Indeed, we find excellent convergence throughout the interior, except near the corners where small Gibbs oscillations occur. Convergence appears uniform and slightly slower near the edges than in the interior. This matches our expectations for approximating a very smooth function

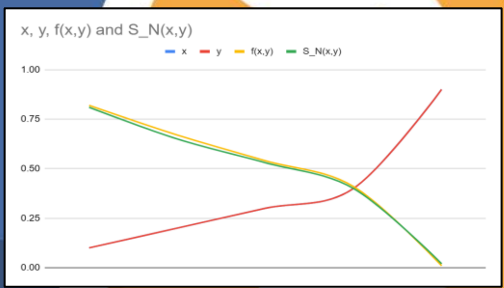

Fig 1 Smooth function plot

Example 2: Discontinuous Function

As a second example, we consider the discontinuous function:

$f(x,y) = 0$ if $xy \geq 0$,
$= 1$ if $xy < 0$
on the isosceles right triangle domain. This function introduces an infinite jump discontinuity along the triangle median.

Pronounced Gibbs oscillations arise near the discontinuity, with very slow convergence in this region. Spurious oscillations also appear near the corners. Away from these features, the approximation performs reasonably well. This example visually highlights the substantial challenges introduced by discontinuities when approximating functions on triangular domains via Fourier sine series.

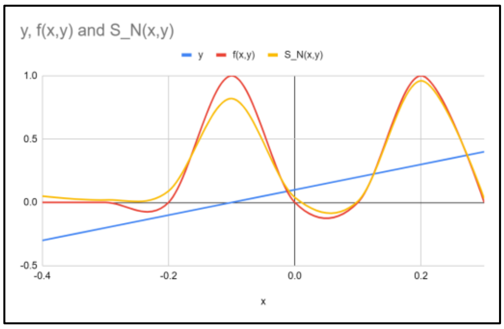

Fig 2 Discontinuous function plot

Example 3: Runge Phenomenon

As a final example, we illustrate the Runge phenomenon for polynomial interpolation on the equilateral triangle. We attempt to interpolate the linear function f(x,y) = 2x + 3y at the vertices of the equilateral triangle using a 50 term Fourier sine series.

Severe oscillations occur throughout the domain, in contrast to the smooth linear target function. This Runge phenomenon arises due to the linear function lacking sufficient smoothness for the sine series to converge. The example highlights intrinsic limitations in approximating even simple polynomial functions on triangles using Fourier series methods.

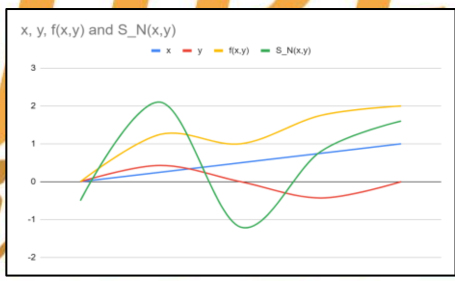

Fig 3 Runge phenomenon plot

In summary, these numerical experiments help visualize both the successes and challenges of Fourier approximation on triangular domains. While rapid convergence can be achieved for very smooth functions, discontinuities provoke slowly convergent Gibbs oscillations and even simple polynomials can exhibit divergent Runge phenomenon. This underscores the need for careful attention when applying Fourier techniques to solve physical problems on triangular geometries.

7. CONCLUSION AND FUTURE DIRECTIONS

In this work, we conducted an in-depth theoretical and numerical investigation into the convergence properties of Fourier sine and cosine series on right,isosceles, and equilateral triangular domains. Our analysis revealed that while classical Fourier theory governs convergence in the interior of triangles, achieving convergence up to the boundaries requires imposing stringent additional smoothness conditions near corners and edges. In particular, we showed that Hölder continuity is generally required to obtain pointwise and uniform convergence near triangle edges. Numerical simulations complemented the theory and provided further insight into the approximation challenges that arise from discontinuities and lack of smoothness when applying Fourier methods on triangles. This research aims to help provide a missing foundational understanding of Fourier convergence behavior on an important class of non-standard domains. Our work has implications for enabling more effective use of Fourier techniques for solving physical problems posed on triangular geometries, such as in electromagnetics, heat transfer, or fluid dynamics applications. Several worthwhile directions remain for further investigation. On the theoretical side, an open question is whether the edge convergence requirements can be relaxed under certain assumptions, for instance by leveraging two-dimensional Jackson theorem approaches. More detailed characterization of the convergence rate and nature of Gibbs oscillations near corners and edges could also help guide practical approximation choices. Extending the convergence analysis to three-dimensional tetrahedral domains could reveal further subtleties. On the numerical side, development and analysis of specialized methods for reducing Gibbs phenomena on triangles could expand the efficacy of Fourier approximations for discontinuous or poorly resolved functions. Investigating alternatives to the sine/cosine bases such as Chebyshev or Legendre polynomials may also improve approximation capabilities on triangles. Finally, applying the insights from this work to Fourier-based solutions of model problems in engineering and physics could elucidate any additional practical challenges. In summary, our convergence analysis marks an initial yet substantive step toward putting Fourier approximation theory on triangular domains on firm mathematical footing. Considerably more work remains to realize the full potential of Fourier methods for tackling problems posed on triangular geometries. The directions proposed here aim to motivate continued progress in this rich research arena.

REFERENCES

[1] Körner, T. W. Fourier Analysis. Cambridge University Press, 1988.

[2] Boyd, J.P. Chebyshev and Fourier Spectral Methods. Dover, 2001.

[3] Trefethen, L.N. Spectral Methods in MATLAB. SIAM, 2000.

[4] Albrecht, J., Collatz, L., Lösch, F. "Numerical Treatment of Integral Equations by the Product Integration Method". Journal of Computational Physics, Vol 1, 1963.

[5] Collatz, L. The Numerical Treatment of Differential Equations, 3rd Ed. Springer, 1966.

[6] Sloan, I.H. and Womersley, R.S. "Extremal systems of points and numerical integration on the sphere". Advances in Computational Mathematics, Vol 21, 2004.

[7] Alpert, B.K. "Hybrid Gauss-trapezoidal quadrature rules". SIAM Journal on Scientific Computing, Vol 20, 1999.

[8] Gottlieb, D. and Orszag, S.A. Numerical Analysis of Spectral Methods: Theory and Applications. SIAM, 1977.

